

ON THE STRUCTURE OF THE WITT GROUP OF BRAIDED FUSION CATEGORIES

ALEXEI DAVYDOV, DMITRI NIKSHYCH, AND VICTOR OSTRIK

1. INTRODUCTION

In our previous work [DMNO] (written jointly with M. Müger) we introduced the Witt group \mathcal{W} of non-degenerate braided fusion categories over an algebraically closed field \mathbb{k} of characteristic zero. Its definition is similar to that of the classical Witt group \mathcal{W}_{pt} of finite abelian groups endowed with a non-degenerate quadratic form with values in \mathbb{k}^\times (such groups correspond to pointed braided fusion categories). Namely, \mathcal{W} is obtained as the quotient of the monoid of non-degenerate braided fusion categories by the submonoid of Drinfeld centers. In particular, \mathcal{W} contains \mathcal{W}_{pt} as a subgroup.

The group \mathcal{W}_{pt} is explicitly known, see Section 2.7. One would like to have a similarly explicit description of the more complicated categorical Witt group \mathcal{W} . In [DMNO] a number of questions regarding the structure of \mathcal{W} was asked. The goal of the present paper is to answer some of these questions and further analyze the structure of \mathcal{W} .

The key observation used in this paper is that one can define the Witt group $s\mathcal{W}$ of *slightly degenerate* braided fusion categories and a group homomorphism $S : \mathcal{W} \rightarrow s\mathcal{W}$ whose kernel is the cyclic group $\mathbb{Z}/16\mathbb{Z}$ generated by the Witt classes of Ising categories, see Section 5.3.

It was pointed to us by V. Drinfeld that completely anisotropic slightly degenerate braided fusion categories admit a canonical decomposition into a tensor product (over the category $s\text{Vec}$ of super vector spaces) of simple categories, see Theorem 4.13. Furthermore, there are very few relations between the classes of simple categories in $s\mathcal{W}$, see Corollary 5.10. This allows to obtain a rather explicit description of the structure of $s\mathcal{W}$ (and, consequently, of \mathcal{W}). Namely, we have

$$(1) \quad s\mathcal{W} = s\mathcal{W}_{pt} \bigoplus s\mathcal{W}_2 \bigoplus s\mathcal{W}_\infty,$$

where $s\mathcal{W}_{pt}$ denotes the subgroup of $s\mathcal{W}$ generated by the Witt classes of slightly degenerate pointed braided fusion categories, $s\mathcal{W}_2$ is an elementary Abelian 2-group, and the subgroup $s\mathcal{W}_\infty$ is a free Abelian group of countable rank, see Proposition 5.18. In particular, groups \mathcal{W} and $s\mathcal{W}$ are 2-primary, i.e., have no odd torsion.

The organization of the present paper can be summarized as follows.

Section 2 contains a necessary background material about fusion categories and definition of the Witt group from [DMNO]. Here we also introduce the notion of tensor product of braided fusion categories over a symmetric category (Section 2.5).

In Section 3 we give a complete classification of étale algebras in tensor products of braided tensor categories. The main result of this Section is Theorem 3.6.

In Section 4 we establish a tensor product decomposition of a slightly degenerate braided fusion category without non-trivial Tannakian subcategories into a tensor product of simple factors. This is done in Theorem 4.13.

Finally, Section 5 contains the definition of the Witt group of non-degenerate braided fusion categories over a symmetric fusion category \mathcal{E} . The principal objects of our study groups \mathcal{W} and $s\mathcal{W}$ correspond to $\mathcal{E} = \text{Vec}$ and $s\text{Vec}$, respectively. Here we describe the structure of $s\mathcal{W}$ (Section 5.4) and describe all relations between the Witt classes of categories $\mathcal{C}(sl(2), k)$, $k \geq 1$.

Acknowledgments We are deeply grateful to Vladimir Drinfeld for sharing with us his ideas regarding the structure of slightly degenerate categories. The statement of Theorem 4.13 is due to him. We are thankful to Michael Müger for collaboration on [DMNO] (of which this paper is a continuation) and for many useful discussions about algebras in tensor products of braided categories that led to results of Section 3. We also thank Terry Gannon for his insights about algebras in braided categories associated to affine Lie algebras. The work of D.N. was partially supported by the NSF grant DMS-0800545. The work of V.O. was partially supported by the NSF grant DMS-0602263.

2. PRELIMINARIES

Throughout this paper our base field \mathbb{k} is an algebraically closed field of characteristic zero.

2.1. Fusion categories. By definition (see [ENO1]), a *fusion category* over \mathbb{k} is a \mathbb{k} -linear semisimple rigid tensor category with finitely many simple objects and finite dimensional spaces of morphisms such that its unit object $\mathbf{1}$ is simple. By a *fusion subcategory* of a fusion category we always mean a full tensor subcategory closed under taking of direct summands. Let Vec denote the fusion category of finite dimensional vector spaces over \mathbb{k} . Any fusion category \mathcal{A} contains a trivial fusion subcategory consisting of multiples of $\mathbf{1}$. We will identify this subcategory with Vec . A fusion category \mathcal{A} is called *simple* if Vec is the only proper fusion subcategory of \mathcal{A} .

If \mathcal{A} and \mathcal{B} are fusion subcategories of a fusion category \mathcal{C} we will denote by $\mathcal{A} \vee \mathcal{B}$ the fusion subcategory of \mathcal{C} generated by \mathcal{A} and \mathcal{B} , that is smallest fusion subcategory which contains both \mathcal{A} and \mathcal{B} .

A fusion category is called *pointed* if all its simple objects are invertible. For a fusion category \mathcal{A} we denote \mathcal{A}_{pt} the maximal pointed fusion subcategory of \mathcal{A} .

We will denote $\mathcal{A} \boxtimes \mathcal{B}$ the tensor product of fusion categories \mathcal{A} and \mathcal{B} .

For a fusion category \mathcal{A} we denote by $\mathcal{O}(\mathcal{A})$ the set of isomorphism classes of simple objects in \mathcal{A} .

Let \mathcal{A} be a fusion category and let $K(\mathcal{A})$ be its Grothendieck ring. There exists a unique ring homomorphism $\text{FPdim} : K(\mathcal{A}) \rightarrow \mathbb{R}$ such that $\text{FPdim}(X) > 0$ for any $0 \neq X \in \mathcal{A}$, see [ENO1, Section 8.1]. For a fusion category \mathcal{A} one defines (see [ENO1, Section 8.2]) its *Frobenius-Perron dimension*:

$$(2) \quad \text{FPdim}(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)^2.$$

Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories such that $\text{FPdim}(\mathcal{A}_1) = \text{FPdim}(\mathcal{A}_2)$. By [EO, Proposition 2.19] a tensor functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an equivalence if and only if it is injective and if and only if it is surjective.

Here we call F *injective* if it is fully faithful and *surjective* if every object Y in \mathcal{A}_2 is a subobject of $F(X)$ for some X in \mathcal{A}_1 .

2.2. Braided fusion categories. A *braided* fusion category is a fusion category \mathcal{C} endowed with a braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, see [JS]. For a braided fusion category its *reverse* \mathcal{C}^{rev} is the same fusion category with a new braiding $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$.

A braided fusion category is *symmetric* if $\tilde{c} = c$. A symmetric fusion category \mathcal{C} is *Tannakian* if there is a finite group G such that \mathcal{C} is equivalent to the category $\text{Rep}(G)$ of finite dimensional representations of G as a braided fusion category. It is known [De] that \mathcal{C} is Tannakian if and only if there is a braided tensor functor $\mathcal{C} \rightarrow \text{Vec}$.

An example of a non-Tannakian symmetric fusion category is the category sVec of super vector spaces.

Recall from [Mu2] that objects X and Y of a braided fusion category \mathcal{C} are said to *centralize* each other if

$$(3) \quad c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$

The *centralizer* \mathcal{D}' of a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is defined to be the full subcategory of objects of \mathcal{C} that centralize each object of \mathcal{D} . It is easy to see that \mathcal{D}' is a fusion subcategory of \mathcal{C} . Clearly, \mathcal{D} is symmetric if and only if $\mathcal{D} \subset \mathcal{D}'$. The *symmetric center* of \mathcal{C} is its self-centralizer \mathcal{C}' .

A braided fusion category \mathcal{C} is called *non-degenerate* if $\mathcal{C}' = \text{Vec}$. If \mathcal{C} has a spherical structure then \mathcal{C} is non-degenerate if and only if it is modular [DGNO1, Proposition 3.7].

For a fusion subcategory \mathcal{D} of a braided fusion category \mathcal{C} one has the following properties (see [DGNO1, Corollary 3.11 and Theorem 3.14]):

$$(4) \quad \mathcal{D}'' = \mathcal{D} \vee \mathcal{C}',$$

$$(5) \quad \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{D} \cap \mathcal{C}').$$

Any pointed braided fusion category comes from a *pre-metric group*, i.e., a finite Abelian group A equipped with a quadratic form $q : A \rightarrow \mathbb{k}^\times$, see [JS, Section 3] and [DGNO1, Section 2.11]. Let $\mathcal{C}(A, q)$ denote the corresponding braided fusion category.

By an *Ising* braided fusion category we understand a non-pointed braided fusion category of the Frobenius-Perron dimension 4. Such a category has 3 classes of simple objects of the Frobenius-Perron dimension 1, 1, and $\sqrt{2}$. An Ising braided fusion category \mathcal{I} is always non-degenerate and $\mathcal{I}_{\text{pt}} \cong \text{sVec}$. It is known that there are 8 equivalence classes of Ising braided fusion categories. See [DGNO1, Appendix B] for a detailed discussion of Ising categories.

The following notion was introduced in [ENO2, Section 2.10]. It plays a crucial role in this paper.

Definition 2.1. A braided fusion category \mathcal{C} is called *slightly degenerate* if $\mathcal{C}' = \text{sVec}$.

Equivalently, \mathcal{C} is slightly degenerate if its symmetric center \mathcal{C}' is non-trivial and contains no non-trivial Tannakian subcategories.

Let \mathcal{C} be a slightly degenerate braided fusion category and let δ denote the simple object generating \mathcal{C}' . Then $\delta \otimes X \not\cong X$ for any simple X in \mathcal{C} , see [Mu1,

Lemma 5.4]. If \mathcal{C} is a pointed slightly degenerate braided fusion category then $\mathcal{C} \cong \mathcal{C}_0 \boxtimes \text{sVec}$, where \mathcal{C}_0 is a non-degenerate braided fusion category, see [ENO2, Proposition 2.6(ii)] or [DGNO1, Corollary A.19].

Example 2.2. There are several general ways to construct examples of slightly degenerate braided fusion categories.

- (i) Let \mathcal{C} be a non-degenerate braided fusion category equipped with a braided tensor functor $\text{sVec} \rightarrow \mathcal{C}$. Then the centralizer of the image of sVec in \mathcal{C} is a slightly degenerate braided fusion category.
- (ii) Let \mathcal{C} be a non-degenerate braided fusion category. Then $\mathcal{C} \boxtimes \text{sVec}$ is a slightly degenerate braided fusion category.

2.3. Drinfeld center of a fusion category. For any fusion category \mathcal{A} its *Drinfeld center* $\mathcal{Z}(\mathcal{A})$ is defined as the category whose objects are pairs (X, γ_X) , where X is an object of \mathcal{A} and $\gamma_X : V \otimes X \simeq X \otimes V$, $V \in \mathcal{A}$, is a natural family of isomorphisms, satisfying well known compatibility conditions. It is known that $\mathcal{Z}(\mathcal{A})$ is a non-degenerate braided fusion category, see [DGNO1, Corollary 3.9]. We have (see [ENO1, Theorem 2.15, Proposition 8.12]):

$$(6) \quad \text{FPdim}(\mathcal{Z}(\mathcal{A})) = \text{FPdim}(\mathcal{A})^2.$$

For a braided fusion category \mathcal{C} there are two braided functors

$$(7) \quad \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{-,X}),$$

$$(8) \quad \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, \tilde{c}_{-,X}).$$

These functors are injective and so we can identify \mathcal{C} and \mathcal{C}^{rev} with their images in $\mathcal{Z}(\mathcal{C})$. These images centralize each other, i.e., $\mathcal{C}' = \mathcal{C}^{\text{rev}}$. This allows to define a braided tensor functor

$$(9) \quad G : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}).$$

It was shown in [Mu3] and [DGNO1, Proposition 3.7] that G is a braided equivalence if and only if \mathcal{C} is non-degenerate.

Let \mathcal{C} be a braided fusion category and let \mathcal{A} be a fusion category. Let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a tensor functor.

Definition 2.3. A structure of a *central functor* on F is a braided tensor functor $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A})$ together with isomorphism between its composition with the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ and F .

Equivalently, a structure of central functor on F is a natural family of isomorphisms $Y \otimes F(X) \xrightarrow{\sim} F(X) \otimes Y$, $X \in \mathcal{C}$, $Y \in \mathcal{A}$, satisfying certain compatibility conditions, see [Be, Section 2.1].

2.4. Étale algebras in braided fusion categories. Here we recall definition and basic properties of étale algebras in braided fusion categories, see [DMNO, Section 3] for details.

Let \mathcal{A} be a fusion category. In this paper an *algebra* $A \in \mathcal{A}$ is an associative algebra with unit, see e.g. [O1, Definition 3.1]. An algebra $A \in \mathcal{A}$ is said to be *separable* if the multiplication morphism $m : A \otimes A \rightarrow A$ splits as a morphism of A -bimodules.

Let \mathcal{C} be a braided fusion category. An algebra A in \mathcal{C} is called *étale* if it is both commutative and separable. We say that an étale algebra $A \in \mathcal{C}$ is *connected* if $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$.

Example 2.4. Let $\mathcal{E} = \text{Rep}(G) \subset \mathcal{C}$ be a Tannakian subcategory of \mathcal{C} . The algebra $\text{Fun}(G)$ of functions on G is a connected étale algebra in \mathcal{C} .

Let A be an étale algebra in \mathcal{C} . An étale algebra *over* A is an étale algebra in \mathcal{C} containing A as a subalgebra. For a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ we denote by $A \cap \mathcal{D}$ the maximal étale subalgebra of A contained in \mathcal{D} .

There is a canonical correspondence between étale algebras in \mathcal{C} and surjective central functors from \mathcal{C} . Namely, let \mathcal{A} a fusion category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a central functor. Let $I : \mathcal{A} \rightarrow \mathcal{C}$ denote the right adjoint functor of F . Then the object $A = I(\mathbf{1}) \in \mathcal{C}$ has a canonical structure of connected étale algebra, see [DMNO, Lemma 3.5].

Conversely, let $A \in \mathcal{C}$ be a connected étale algebra. Let \mathcal{C}_A denote the category of right A -modules in \mathcal{C} . Then the free module functor

$$F_A : \mathcal{C} \rightarrow \mathcal{C}_A, \quad X \mapsto X \otimes A$$

is surjective and has a canonical structure of a central functor. The above constructions (of étale algebras from central functors and vice versa) are inverses of each other. It was shown in [DMNO, Lemma 3.11] that

$$(10) \quad \text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}.$$

Note that every right A -module M can be made an A -bimodule in two ways, using the braiding c of \mathcal{C} and its reverse \tilde{c} . We call M *dyslectic* (or *local*) if two resulting bimodules are equal. The category \mathcal{C}_A^0 of dyslectic A -modules has a canonical structure of a braided fusion category. If \mathcal{C} is non-degenerate then so is \mathcal{C}_A^0 and we have

$$(11) \quad \text{FPdim}(\mathcal{C}_A^0) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)^2},$$

see [DMNO, Corollary 3.32].

A *Lagrangian algebra* in a non-degenerate braided fusion category \mathcal{C} is a connected étale algebra $A \in \mathcal{C}$ such that $\text{FPdim}(\mathcal{C}) = \text{FPdim}(A)^2$, see [DMNO, Definition 4.6]. A Lagrangian algebra in \mathcal{C} exists if and only if $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$ for some fusion category \mathcal{A} . Moreover, the isomorphism classes of Lagrangian algebras in $\mathcal{Z}(\mathcal{A})$ are in bijection with equivalence classes of indecomposable \mathcal{A} -module categories, see [KR, Theorem 1.1] and [DMNO, Proposition 4.8].

Following [DMNO] we say that a braided fusion category \mathcal{C} is *completely anisotropic* if the only connected étale algebra $A \in \mathcal{C}$ is $A = \mathbf{1}$. By Example 2.4, a completely anisotropic category has no Tannakian subcategories other than Vec and so is either non-degenerate or slightly degenerate. By [DMNO, Lemma 5.12] any central functor from a completely anisotropic braided tensor category is injective.

2.5. Fusion categories over a symmetric category. The following definition was given in [DGNO1, Definition 4.16].

Definition 2.5. Let \mathcal{E} be a symmetric fusion category. A *fusion category over* \mathcal{E} is a fusion category \mathcal{A} equipped with a braided tensor functor $T : \mathcal{E} \rightarrow \mathcal{Z}(\mathcal{A})$.

In this paper we will only consider fusion categories \mathcal{A} over \mathcal{E} with the property that the composition of T with the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ is fully faithful. Note that this property is automatically satisfied for $\mathcal{E} = \text{sVec}$.

In other words, a structure of a tensor functor over \mathcal{E} on $F : \mathcal{A} \rightarrow \mathcal{B}$ is a tensor isomorphism $u_E : F(T_{\mathcal{A}}(E)) \rightarrow T_{\mathcal{B}}(E)$, $E \in \mathcal{E}$, such that the diagram

$$\begin{array}{ccccc} F(T_{\mathcal{A}}(E) \otimes X) & \xrightarrow{F_{T(E), X}} & F(T_{\mathcal{A}}(E)) \otimes F(X) & \xrightarrow{u_X} & T_{\mathcal{B}}(E) \otimes F(X) \\ \downarrow & & & & \downarrow \\ F(X \otimes T_{\mathcal{A}}(E)) & \xrightarrow{F_{X, T(E)}} & F(X) \otimes F(T_{\mathcal{A}}(E)) & \xrightarrow{u_X} & F(X) \otimes T_{\mathcal{B}}(E) \end{array}$$

commutes. Here the vertical arrows are the central structures of $T_{\mathcal{A}}(E)$ and $T_{\mathcal{B}}(E)$, respectively.

A tensor natural transformation $a : F \rightarrow G$ between tensor functors over \mathcal{E} is a *tensor natural transformation over \mathcal{E}* if the diagram

$$\begin{array}{ccc} F(T_{\mathcal{A}}(E)) & \xrightarrow{a_{T(E)}} & G(T_{\mathcal{A}}(E)) \\ & \searrow u_E & \swarrow v_E \\ & T_{\mathcal{B}}(E) & \end{array}$$

commutes for any $E \in \mathcal{E}$. Here u_E and v_E denote the structures of functors over \mathcal{E} for F and G , respectively.

Note that relative centers of successive tensor functors $F : \mathcal{A} \rightarrow \mathcal{B}, G : \mathcal{B} \rightarrow \mathcal{C}$ are related to the relative center of the composition by a pair of canonical tensor functors $\mathcal{Z}(F) \rightarrow \mathcal{Z}(G \circ F), \mathcal{Z}(G) \rightarrow \mathcal{Z}(G \circ F)$ fitting into a commutative diagram of tensor functors:

$$\begin{array}{ccccc} & & \mathcal{Z}(G \circ F) & & \\ & \nearrow & & \searrow & \\ \mathcal{Z}(F) & & & & \mathcal{Z}(G) \\ \nearrow & \searrow & & \nearrow & \searrow \\ \mathcal{Z}(\mathcal{A}) & & \mathcal{Z}(\mathcal{B}) & & \mathcal{Z}(\mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array}$$

This allows us to compose tensor functors over \mathcal{E} . In other words, the structure of tensor functor over \mathcal{E} on the composition $G \circ F$ is given by

$$G(F(T_{\mathcal{A}}(E))) \xrightarrow{G(u_E)} G(T_{\mathcal{B}}(E)) \xrightarrow{v_E} T_{\mathcal{C}}(E)$$

Thus we have a 2-category $\mathbf{Fus}_{\mathcal{E}}$ of fusion categories, functors and natural transformations over a symmetric fusion category \mathcal{E} .

Let us explain the functoriality of 2-categories $\mathbf{Fus}_{\mathcal{E}}$ with respect to braided tensor functors $\mathcal{E} \rightarrow \mathcal{F}$. Construct a *base change* 2-functor

$$(12) \quad - \boxtimes_{\mathcal{E}} \mathcal{F} : \mathbf{Fus}_{\mathcal{E}} \rightarrow \mathbf{Fus}_{\mathcal{F}}, \quad \mathcal{A} \mapsto \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{F}.$$

Here the tensor product $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{F}$ of fusion categories over \mathcal{E} is defined as follows. Consider a composition of braided tensor functors

$$\mathcal{E} \boxtimes \mathcal{F} \longrightarrow \mathcal{F} \boxtimes \mathcal{F} \xrightarrow{\otimes} \mathcal{F}$$

where the first is given by the braided functor $\mathcal{E} \rightarrow \mathcal{F}$ and the second is the tensor product of \mathcal{F} . Denote by R the right adjoint to the composition. The value $R(\mathbf{1})$ on the unit object is a connected étale algebra in $\mathcal{E} \boxtimes \mathcal{F}$. Denote by A its image in $\mathcal{A} \boxtimes \mathcal{F}$. Now $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{F}$ is the category $(\mathcal{A} \boxtimes \mathcal{F})_A$ of A -modules in $\mathcal{A} \boxtimes \mathcal{F}$.

Note that Deligne tensor product of fusion categories induces a 2-functor

$$\mathbf{Fus}_{\mathcal{E}} \times \mathbf{Fus}_{\mathcal{F}} \rightarrow \mathbf{Fus}_{\mathcal{E} \boxtimes \mathcal{F}}, \quad (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \boxtimes \mathcal{B}.$$

Assume that $\mathcal{F} = \mathcal{E}$ and compose the above 2-functor with the base change 2-functor

$$- \boxtimes_{\mathcal{E} \boxtimes \mathcal{E}} \mathcal{E} : \mathbf{Fus}_{\mathcal{E} \boxtimes \mathcal{E}} \rightarrow \mathbf{Fus}_{\mathcal{E}}$$

induced by the tensor product functor $\mathcal{E} \boxtimes \mathcal{E} \rightarrow \mathcal{E}$ (which is a braided monoidal functor). This gives a 2-functor (the *Deligne tensor product over \mathcal{E}*)

$$\mathbf{Fus}_{\mathcal{E}} \times \mathbf{Fus}_{\mathcal{E}} \rightarrow \mathbf{Fus}_{\mathcal{E}}, \quad (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}$$

which turns $\mathbf{Fus}_{\mathcal{E}}$ into a monoidal 2-category.

Remark 2.8. More explicitly, we define $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}$ as the category of (right) A -modules $(\mathcal{A} \boxtimes \mathcal{B})_A$, where A is an étale algebra in $\mathcal{Z}(\mathcal{A} \boxtimes \mathcal{B})$ defined as $(T_{\mathcal{A}} \boxtimes T_{\mathcal{B}})(R(\mathbf{1}))$, with $R : \mathcal{E} \rightarrow \mathcal{E} \boxtimes \mathcal{E}$ being the right adjoint functor to the tensor product functor $\otimes : \mathcal{E} \boxtimes \mathcal{E} \rightarrow \mathcal{E}$.

Note that our construction gives the tensor product of fusion categories over a symmetric category defined by Greenough in [G, Section 6].

The base change 2-functors preserve tensor product, i.e., are monoidal:

$$(\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}) \boxtimes_{\mathcal{E}} \mathcal{F} \simeq (\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{F}) \boxtimes_{\mathcal{F}} (\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{F}).$$

Definition 2.9. A *braided fusion category over \mathcal{E}* or *braided fusion \mathcal{E} -category* is a braided fusion category \mathcal{C} equipped with a braided tensor embedding $T : \mathcal{E} \rightarrow \mathcal{C}'$. An \mathcal{E} -subcategory of a braided fusion \mathcal{E} -category is its fusion subcategory containing \mathcal{E} .

A braided fusion category over \mathcal{E} can be seen as a fusion category over \mathcal{E} with a braided functor $T_{\mathcal{C}} : \mathcal{E} \rightarrow \mathcal{Z}(\mathcal{C})$ defined as the composition of braided functors

$$\mathcal{E} \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{Z}(\mathcal{C})$$

Let \mathcal{C} and \mathcal{D} be braided fusion categories over \mathcal{E} . Since the functors $T_{\mathcal{C}}, T_{\mathcal{D}}$ factor through symmetric centers $\mathcal{C}', \mathcal{D}'$ respectively the algebra $A = (T_{\mathcal{C}} \boxtimes T_{\mathcal{D}})(R(\mathbf{1}))$ is an object of the symmetric center $(\mathcal{C} \boxtimes \mathcal{D})'$. Thus the category of modules $(\mathcal{A} \boxtimes \mathcal{B})_A$ coincides with its subcategory of local modules $(\mathcal{A} \boxtimes \mathcal{B})_A^0$ and hence is braided. In other words the Deligne tensor product $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ of two braided fusion categories over \mathcal{E} is a braided fusion category over \mathcal{E} .

Remark 2.10. Recall from [JS, section 5] that a braiding on a fusion category \mathcal{C} is equivalent to a monoidal structure on the tensor product functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$. In slightly more general and abstract terms, the 2-category \mathbf{Fus}^{br} of braided fusion categories is bi-equivalent to the 2-category of *pseudo-monoids* in \mathbf{Fus} .

The above statement can be generalized to fusion categories over \mathcal{E} . It is not hard to see that the structure of a braided fusion category over \mathcal{E} on \mathcal{C} is equivalent to a monoidal structure on the tensor product functor $\otimes : \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{C} \rightarrow \mathcal{C}$. Again reformulating it in slightly more general and abstract terms, the 2-category $\mathbf{Fus}_{\mathcal{E}}^{\text{br}}$

of braided fusion categories is bi-equivalent to the 2-category of pseudo-monoids in $\mathbf{Fus}_{\mathcal{E}}$.

This in particular allows us to extend base change to braided fusion categories over \mathcal{E} . More precisely, for a braided fusion category \mathcal{C} over a symmetric \mathcal{E} and for a symmetric tensor functor $\mathcal{E} \rightarrow \mathcal{F}$ the base change $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{F}$ has a natural structure of a braided fusion category over \mathcal{F} .

2.6. De-equivariantization. Let \mathcal{A} be a fusion category and let $\mathcal{E} = \text{Rep}(G)$ be a Tannakian subcategory of $\mathcal{Z}(\mathcal{A})$ which embeds into \mathcal{A} via the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. Then \mathcal{A} is a fusion category over \mathcal{E} . The *de-equivariantization* of \mathcal{A} is the fusion category $\mathcal{A} \boxtimes_{\mathcal{E}} \text{Vec}$ obtained from \mathcal{A} by means of the base change (12) corresponding to the braided fiber functor $\mathcal{E} \rightarrow \text{Vec}$.

Explicitly, $\mathcal{A} \boxtimes_{\mathcal{E}} \text{Vec} = \text{Rep}_{\mathcal{A}}(A)$ where $A = \text{Fun}(G)$ is the regular algebra of \mathcal{E} (viewed as an étale algebra in $\mathcal{Z}(\mathcal{A})$), see Example 2.4 and $\text{Rep}_{\mathcal{A}}(A)$ is defined in [DMNO, Definition 3.16].

There is a canonical surjective tensor functor $\mathcal{A} \rightarrow \mathcal{A} \boxtimes_{\mathcal{E}} \text{Vec}$ assigning to each $X \in \mathcal{A}$ the free A -module $X \otimes A$.

The above construction extends to braided fusion categories as follows. Let $\mathcal{E} \subset \mathcal{C}'$ be a Tannakian subcategory. Then \mathcal{C} is a braided fusion category over \mathcal{E} , its de-equivariantization $\mathcal{C} \boxtimes_{\mathcal{E}} \text{Vec}$ is a braided fusion category, and the canonical central tensor functor $\mathcal{C} \rightarrow \mathcal{C} \boxtimes_{\mathcal{E}} \text{Vec}$ is braided.

We refer the reader to [DGNO1, Section 4] for details.

Remark 2.11. Tensor product of fusion categories \mathcal{A} and \mathcal{B} over a symmetric category \mathcal{E} defined in Section 2.5 is a special case of de-equivariantization. Indeed, let \mathcal{F} be the Tannakian subcategory of $\mathcal{E} \boxtimes \mathcal{E}$ such that $(\mathcal{E} \boxtimes \mathcal{E}) \boxtimes_{\mathcal{F}} \text{Vec} \cong \mathcal{E}$ (existence of such a subcategory \mathcal{F} follows from [DMNO, Corollary 3.22] since the tensor product functor $\otimes : \mathcal{E} \boxtimes \mathcal{E} \rightarrow \mathcal{E}$ is braided). Then

$$\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \cong (\mathcal{A} \boxtimes \mathcal{B}) \boxtimes_{\mathcal{F}} \text{Vec}.$$

The étale algebra $A \in \mathcal{Z}(\mathcal{A} \boxtimes \mathcal{B})$ constructed in Remark 2.10 identifies with the regular algebra of \mathcal{F} .

Proposition 2.12. *Let \mathcal{A} be a fusion category over a Tannakian category \mathcal{E} such that the functor $\mathcal{E} \rightarrow \mathcal{Z}(\mathcal{A})$ is an embedding. Then $\mathcal{Z}(\mathcal{A} \boxtimes_{\mathcal{E}} \text{Vec}) \cong \mathcal{E}' \boxtimes_{\mathcal{E}} \text{Vec}$. (Here we identify \mathcal{E} with its image in $\mathcal{Z}(\mathcal{A})$).*

Proposition 2.12 is proved in [ENO2, Proposition 2.10].

Proposition 2.13. *Let \mathcal{C} be a braided fusion category, let $\mathcal{E} \subset \mathcal{C}'$ be a Tannakian subcategory, and let $\mathcal{D} \subset \mathcal{C}$ be a fusion subcategory containing \mathcal{E} . The braided fusion category $\mathcal{D}' \boxtimes_{\mathcal{E}} \text{Vec}$ is equivalent to the centralizer of $\mathcal{D} \boxtimes_{\mathcal{E}} \text{Vec}$ in $\mathcal{C} \boxtimes_{\mathcal{E}} \text{Vec}$. In other words, de-equivariantization of braided fusion categories commutes with taking the centralizers.*

Proposition 2.13 is proved in [DGNO1, Proposition 4.30].

2.7. The Witt group of non-degenerate braided fusion categories. The *Witt group* \mathcal{W} of non-degenerate braided fusion categories was defined in [DMNO].

Two non-degenerate braided fusion categories \mathcal{C}_1 and \mathcal{C}_2 are *Witt equivalent* if there exist fusion categories \mathcal{A}_1 and \mathcal{A}_2 such that $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \cong \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$. The elements of \mathcal{W} are Witt equivalence classes of non-degenerate braided fusion

categories. The group operation of \mathcal{W} is given by the Deligne tensor product \boxtimes . Let $[\mathcal{C}]$ denote the Witt equivalence class containing category \mathcal{C} . The unit object of \mathcal{W} is $[\text{Vec}]$ and the inverse of $[\mathcal{C}]$ is $[\mathcal{C}^{\text{rev}}]$.

The following results were established in [DMNO, Section 5]. For any étale algebra $A \in \mathcal{C}$ we have $[\mathcal{C}] = [\mathcal{C}_A^0]$. The Witt class of \mathcal{C} is trivial, i.e., $[\mathcal{C}] = [\text{Vec}]$, if and only if $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$ for some fusion category \mathcal{A} . Every Witt class contains a unique (up to equivalence) completely anisotropic representative.

The Witt group \mathcal{W} contains the subgroup \mathcal{W}_{pt} of the Witt classes of non-degenerate pointed braided fusion categories [DMNO, Section 5.3]. The latter coincides with the classical Witt group of metric groups (i.e., finite abelian groups equipped with a non-degenerate quadratic form). The group \mathcal{W}_{pt} is explicitly known, see e.g., [DGNO1, Appendix A.7]. Namely,

$$(13) \quad \mathcal{W}_{pt} = \bigoplus_{p \text{ is prime}} \mathcal{W}_{pt}(p),$$

where $\mathcal{W}_{pt}(p) \subset \mathcal{W}_{pt}$ consists of the classes of metric p -groups.

The group $\mathcal{W}_{pt}(2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by two classes $[\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_1)]$ and $[\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q_2)]$, where q_1, q_2 are any non-degenerate forms. For $p \equiv 3 \pmod{4}$ we have $\mathcal{W}_{pt}(p) \cong \mathbb{Z}/4\mathbb{Z}$ and the class $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q)]$ is a generator for any non-degenerate form q . For $p \equiv 1 \pmod{4}$ the group $\mathcal{W}_{pt}(p)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by the two classes $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q')]$ and $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q'')]$ with $q'(l) = \zeta^{l^2}$ and $q''(l) = \zeta^{nl^2}$, where ζ is a primitive p th root of unity in \mathbb{k} and n is any quadratic non-residue modulo p .

It was also explained in [DMNO, Section 6.4] that \mathcal{W} contains a cyclic subgroup \mathcal{W}_{Ising} of order 16 generated by the Witt classes of Ising braided fusion categories. This group can explicitly be described as follows. For every Ising braided category \mathcal{I} the class $[\mathcal{I}]$ is a generator of \mathcal{W}_{Ising} . The unique index 2 subgroup of \mathcal{W}_{Ising} consists of the Witt classes of categories $\mathcal{C}(A, q)$, where (A, q) is a metric group of order 4 such that there exists $u \in A$ with $q(u) = -1$ (cf. [DGNO1, §A.3.2]).

2.8. Tensor categories from affine Lie algebras. Let \mathfrak{g} be a finite dimensional simple Lie algebra and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. For any $k \in \mathbb{Z}_{>0}$ let $\mathcal{C}(\mathfrak{g}, k)$ be the category of highest weight integrable $\hat{\mathfrak{g}}$ -modules of level k , see e.g. [BaKi, Section 7.1] where this category is denoted \mathcal{O}_k^{int} . The category $\mathcal{C}(\mathfrak{g}, k)$ can be identified with the category $\text{Rep}(V(\mathfrak{g}, k))$, where $V(\mathfrak{g}, k)$ is the simple vertex operator algebra (VOA) associated with the vacuum $\hat{\mathfrak{g}}$ -module of level k , the *affine VOA*. Thus the category $\mathcal{C}(\mathfrak{g}, k)$ has a structure of modular tensor category, see [HuL], [BaKi, Chapter 7]. In particular the category $\mathcal{C}(\mathfrak{g}, k)$ is braided and non-degenerate.

Example 2.14. The category $\mathcal{C}(sl(n), 1)$ is pointed. It identifies with $\mathcal{C}(\mathbb{Z}/n\mathbb{Z}, q)$ where $q(l) = e^{\pi i l^2 \frac{n-1}{n}}$, $l \in \mathbb{Z}/n\mathbb{Z}$. More generally, for a simply laced \mathfrak{g} the category $\mathcal{C}(\mathfrak{g}, 1)$ is pointed [FK].

The following formula for the central charge is very useful, see e.g. [BaKi, 7.4.5]:

$$(14) \quad c(\mathcal{C}(\mathfrak{g}, k)) = \frac{k \dim \mathfrak{g}}{k + h^\vee},$$

where h^\vee is the dual Coxeter number of the Lie algebra \mathfrak{g} .

Here we collect some basic facts about categories $\mathcal{C}(sl(2), k)$.

Simple objects $[j]$ of $\mathcal{C}(sl(2), k)$ are labelled by integers $j = 0, \dots, k$. The decomposition of tensor products of simple objects is given by

$$[i] \otimes [j] = \begin{cases} \bigoplus_{s=0}^{\min(i,j)} [i+j-2s], & i+j < k \\ \bigoplus_{s=i+j-k}^{\min(i,j)} [i+j-2s], & i+j \geq k \end{cases}$$

There is a canonical ribbon structure on $\mathcal{C}(sl(2), k)$:

$$\theta_{[j]} = e^{2\pi i \frac{j(j+2)}{4(k+2)}} \text{id}_{[j]},$$

which gives the square of the braiding:

$$c_{[j],[t]} c_{[t],[j]} = \bigoplus_s e^{2\pi (h_{t+j-2s} - h_t - h_j)} \text{id}_{[t+j-2s]},$$

where $h_j = \frac{j(j+2)}{4(k+2)}$. The Frobenius-Perron dimensions of simple objects are

$$\text{FPdim}([j]) = \frac{q^{j+1} - q^{-j-1}}{q - q^{-1}}, \quad q = e^{\frac{\pi i}{k+2}}.$$

The Frobenius-Perron dimension of $\mathcal{C}(sl(2), k)$ is

$$\text{FPdim}(\mathcal{C}(sl(2), k)) = \frac{k+2}{2 \sin^2(\frac{\pi i}{k+2})}.$$

The multiplicative central charge of $\mathcal{C}(sl(2), k)$ is

$$\xi(\mathcal{C}(sl(2), k)) = e^{2\pi i \frac{3k}{8(k+2)}}.$$

3. ÉTALE ALGEBRAS IN TENSOR PRODUCTS OF BRAIDED FUSION CATEGORIES

Let \mathcal{C} be a braided fusion category.

3.1. Étale subalgebras.

Lemma 3.1. *Let A be an étale algebra in \mathcal{C} . There is a bijection between étale algebras over A and étale algebras in \mathcal{C}_A^0 .*

Proof. This statement was proved in [DMNO, Proposition 3.16]. See also [FFRS, Lemma 4.13] and [Da, Proposition 2.3.3]. \square

Lemma 3.2. *Let $A \in \mathcal{C}$ be a separable algebra and let $\mathcal{D} \subset \mathcal{C}$ be a fusion subcategory. Then $A \cap \mathcal{D}$ is separable.*

Proof. Let us consider the category \mathcal{C}_A as a module category over \mathcal{D} . It is clear from definitions that $\underline{\text{Hom}}_{\mathcal{D}}(A, A) = A \cap \mathcal{D}$, where $\underline{\text{Hom}}_{\mathcal{D}}$ denotes the internal Hom in \mathcal{D} . Hence, the category $\mathcal{D}_{A \cap \mathcal{D}}$ coincides with \mathcal{D} -module subcategory of \mathcal{C}_A generated by $A \in \mathcal{C}_A$, see [O1, Section 3.3]; in particular this category is semisimple. This completes the proof. \square

Corollary 3.3. *Assume \mathcal{C} is a braided fusion category. Let $A \in \mathcal{C}$ be an étale algebra and let $\mathcal{D} \subset \mathcal{C}$ be a fusion subcategory. Then $A \cap \mathcal{D}$ is étale.* \square

Remark 3.4. An interesting open question is whether *any* subalgebra of an étale algebra is étale.

3.2. Classification of étale algebras in tensor products. Let \mathcal{C} be a braided fusion category. Let $G : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C})$ be the canonical braided tensor functor, see (9). The tensor product functor $\otimes : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$ factors through G and so has a natural structure of central functor. Let $I_{\otimes} : \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ be the right adjoint functor of \otimes . The object $I_{\otimes}(\mathbf{1}) \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ has a structure of étale algebra, see Section 2.4.

It is easy to compute that $I_{\otimes}(\mathbf{1}) = \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X^* \boxtimes X$. Thus, for a braided fusion category \mathcal{C} the object $\bigoplus_{X \in \mathcal{O}(\mathcal{C})} X^* \boxtimes X \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ has a canonical structure of connected étale algebra.

We now generalize this construction as follows. Let \mathcal{C} and \mathcal{D} be two braided fusion categories and let $A_1 \in \mathcal{C}$ and $A_2 \in \mathcal{D}$ be two connected étale algebras. Let $\mathcal{C}_1 \subset \mathcal{C}_{A_1}^0$ and $\mathcal{D}_1 \subset \mathcal{D}_{A_2}^0$ be two fusion subcategories, and let $\phi : \mathcal{C}_1^{\text{rev}} \simeq \mathcal{D}_1$ be a braided equivalence. Using tensor functors

$$\mathcal{C}_1 \boxtimes \mathcal{C}_1^{\text{rev}} \xrightarrow{\text{id} \boxtimes \phi} \mathcal{C}_1 \boxtimes \mathcal{D}_1 \hookrightarrow \mathcal{C}_{A_1}^0 \boxtimes \mathcal{D}_{A_2}^0 = (\mathcal{C} \boxtimes \mathcal{D})_{A_1 \boxtimes A_2}^0$$

and Lemma 3.1 we can consider the algebra $I_{\otimes}(\mathbf{1}) \in \mathcal{C}_1 \boxtimes \mathcal{C}_1^{\text{rev}}$ as an étale algebra over $A_1 \boxtimes A_2$ in $\mathcal{C} \boxtimes \mathcal{D}$. We will call this algebra $A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)$.

Lemma 3.5. *We have*

$$\text{FPdim}(A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)) = \text{FPdim}(A_1) \text{FPdim}(A_2) \text{FPdim}(\mathcal{C}_1).$$

Proof. For an object $M \in \mathcal{C}_{A_1}$ we have two possible Frobenius-Perron dimensions: $\text{FPdim}(M)$ where M is considered as an object of fusion category \mathcal{C}_{A_1} and $\text{FPdim}_{\mathcal{C}}(M)$ where M is considered as an object of fusion category \mathcal{C} . It is a straightforward consequence of [ENO1, Proposition 8.7] that

$$\text{FPdim}_{\mathcal{C}}(M) = \text{FPdim}(M) \text{FPdim}(A_1).$$

We have:

$$\begin{aligned} \text{FPdim}(A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)) &= \sum_{X \in \mathcal{O}(\mathcal{C}_1)} \text{FPdim}_{\mathcal{C}}(X^*) \text{FPdim}_{\mathcal{D}}(\phi(X)) \\ &= \sum_{X \in \mathcal{O}(\mathcal{C}_1)} \text{FPdim}(A_1) \text{FPdim}(X)^2 \text{FPdim}(A_2) \\ &= \text{FPdim}(A_1) \text{FPdim}(A_2) \text{FPdim}(\mathcal{C}_1). \end{aligned}$$

□

Here is the main result of this section:

Theorem 3.6. *The algebras $A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)$ are pairwise non-isomorphic and any connected étale algebra in $\mathcal{C} \boxtimes \mathcal{D}$ is isomorphic to one of them.*

Proof. Let A be an étale algebra in $\mathcal{C} \boxtimes \mathcal{D}$. We will construct data $A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi$ such that $A \simeq A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)$. Moreover, if the constructions below applied to $A = A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)$ we recover the original $A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi$.

Set $A_1 = A \cap (\mathcal{C} \boxtimes \mathbf{1})$ and $A_2 = A \cap (\mathbf{1} \boxtimes \mathcal{D})$. By Corollary 3.3 the algebras $A_1 \in \mathcal{C}, A_2 \in \mathcal{D}$ and $A_1 \boxtimes A_2 \in \mathcal{C} \boxtimes \mathcal{D}$ are connected étale. The algebra A can be considered as an étale algebra in $(\mathcal{C} \boxtimes \mathcal{D})_{A_1 \boxtimes A_2}^0 \simeq \mathcal{C}_{A_1}^0 \boxtimes \mathcal{D}_{A_2}^0$. Replacing \mathcal{C} by $\mathcal{C}_{A_1}^0$ and \mathcal{D} by $\mathcal{D}_{A_2}^0$ we reduce the Theorem to the case when $A_1 = \mathbf{1}$ and $A_2 = \mathbf{1}$.

Let $A \in \mathcal{C} \boxtimes \mathcal{D}$ be an étale algebra such that $A \cap (\mathcal{C} \boxtimes \mathbf{1}) = A \cap (\mathbf{1} \boxtimes \mathcal{D}) = \mathbf{1}$. Consider fusion category $\mathcal{A} = (\mathcal{C} \boxtimes \mathcal{D})_A$. The restrictions of the canonical central

functor $\mathcal{C} \boxtimes \mathcal{D} \rightarrow (\mathcal{C} \boxtimes \mathcal{D})_A$ to $\mathcal{C} = \mathcal{C} \boxtimes \mathbf{1}$ and $\mathcal{D} = \mathbf{1} \boxtimes \mathcal{D}$ are injective. Let $\mathcal{A}_1 \subset \mathcal{A}$ be the intersection of the images of \mathcal{C} and \mathcal{D} in \mathcal{A} . Then there are fusion subcategories $\mathcal{C}_1 \subset \mathcal{C}$ and $\mathcal{D}_1 \subset \mathcal{D}$ such that the functors above restrict to equivalences $\mathcal{C}_1 \simeq \mathcal{A}_1$ and $\mathcal{D}_1 \simeq \mathcal{A}_1$. Combining these equivalences we get a tensor equivalence $\phi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{D}_1$; it is clear that the algebra A identifies with $\bigoplus_{X \in \mathcal{O}(\mathcal{C}_1)} X^* \boxtimes \phi(X)$ (more precisely, it identifies with the image of $I_{\otimes}(\mathbf{1}) \in \mathcal{C}_1 \boxtimes \mathcal{C}_1^{\text{rev}}$ under the equivalence $\text{id} \boxtimes \phi$).

To finish the proof we need to show that the equivalence $\phi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{D}_1$ constructed above is *braided* when considered as a functor $\mathcal{C}_1^{\text{rev}} \rightarrow \mathcal{D}_1$. For this we can assume that that $\mathcal{C} = \mathcal{C}_1$ and $\mathcal{D} = \mathcal{D}_1$ (and so $\mathcal{A} = \mathcal{A}_1$). The functor $\mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{A}$ is central, i.e., it factorizes as $\mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. The functors $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A})$ and $\mathcal{D} \rightarrow \mathcal{Z}(\mathcal{A})$ are injective; if we identify \mathcal{C} and \mathcal{D} with their images in $\mathcal{Z}(\mathcal{A})$, then they centralize each other, that is $\mathcal{D} \subset \mathcal{C}'_{\mathcal{Z}(\mathcal{A})}$, where $\mathcal{C}'_{\mathcal{Z}(\mathcal{A})}$ denotes the centralizer of \mathcal{C} in $\mathcal{Z}(\mathcal{A})$. Since $\text{FPdim}(\mathcal{Z}(\mathcal{A})) = \text{FPdim}(\mathcal{A})^2$ and $\text{FPdim}(\mathcal{A}) = \text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D})$, we see from by (5) that $\text{FPdim}(\mathcal{C}'_{\mathcal{Z}(\mathcal{A})}) = \text{FPdim}(\mathcal{D})$, therefore $\mathcal{D} = \mathcal{C}'_{\mathcal{Z}(\mathcal{A})}$. On the other hand, since $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ is an equivalence, we have $\mathcal{C}'_{\mathcal{Z}(\mathcal{A})} \cong \mathcal{C}^{\text{rev}} \subset \mathcal{Z}(\mathcal{A})$. It follows from definitions that the functor ϕ is isomorphic to the composition of braided tensor functors $\mathcal{C}^{\text{rev}} \xrightarrow{\sim} \mathcal{C}'_{\mathcal{Z}(\mathcal{A})} \xrightarrow{\sim} \mathcal{D}$ and, hence, it is braided. \square

3.3. Lagrangian algebras in the center of non-degenerate braided fusion category. Let \mathcal{C} be a non-degenerate braided fusion category. We introduced the notion of a Lagrangian algebra in \mathcal{C} in Section 2.4.

Proposition 3.7. *Let \mathcal{C} be a non-degenerate braided fusion category. A Lagrangian algebra in $\mathcal{Z}(\mathcal{C})$ is of the form $A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)$ with $\text{FPdim}(A_1) = \text{FPdim}(A_2)$, $\mathcal{C}_1 = \mathcal{C}_{A_1}^0$, $\mathcal{D}_1 = (\mathcal{C}^{\text{rev}})_{A_2}^0$.*

Proof. Since \mathcal{C} is non-degenerate, we have $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$, see Section 2.2. Using Lemma 3.5 and (11) we have

$$\begin{aligned} \text{FPdim}(A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)) &\leq \text{FPdim}(A_1)\text{FPdim}(A_2)\text{FPdim}(\mathcal{C}_{A_1}^0) \\ &= \frac{\text{FPdim}(A_2)}{\text{FPdim}(A_1)}\text{FPdim}(\mathcal{C}) \end{aligned}$$

and, similarly,

$$\text{FPdim}(A(A_1, A_2, \mathcal{C}_1, \mathcal{D}_1, \phi)) \leq \frac{\text{FPdim}(A_1)}{\text{FPdim}(A_2)}\text{FPdim}(\mathcal{C}).$$

The result follows. \square

Corollary 3.8. *Let \mathcal{C} be a non-degenerate braided fusion category. Equivalence classes of indecomposable module categories over \mathcal{C} are parameterized by isomorphism classes of triples (A_1, A_2, ϕ) where $A_1, A_2 \in \mathcal{C}$ are connected étale algebras and $\phi : \mathcal{C}_{A_1}^0 \xrightarrow{\sim} (\mathcal{C}_{A_2}^0)^{\text{rev}}$ is a braided equivalence.*

Proof. This statement follows by combining Proposition 3.7 and the bijection between module categories over a fusion category and Lagrangian algebras in its center, see Section 2.4. \square

Remark 3.9. The case $A_1 = A_2 = \mathbf{1}$ corresponds to the *invertible* module categories over \mathcal{C} , see [ENO3, §5.4]. The result above shows that the invertible module categories over \mathcal{C} are in bijection with braided autoequivalences of \mathcal{C} . This is a weak form of [ENO3, Theorem 5.2].

Remark 3.10. In the setup of Rational Conformal Field Theory, the category \mathcal{C} is the representation category of a vertex algebra and a Lagrangian algebra $A \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ (or rather the underlying vector space) is the Hilbert space of physical states; the class $[A] \in K(\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}) = K(\mathcal{C}) \otimes_{\mathbb{Z}} K(\mathcal{C})$ is the *modular invariant* (or partition function) of the theory. In the physical terminology, the modular invariant is of type I if $A_1 = A_2$ and $\phi = \text{id}$; of type II if $A_1 = A_2$ but $\phi \not\simeq \text{id}$; and is *heterotic* if $A_1 \not\simeq A_2$ (one should be careful, sometimes the same terminology is used when A_1, A_2, ϕ are replaced by their classes in the Grothendieck group). In this language, Proposition 3.7 says that each modular invariant has two type I “parents”. This result was initially observed by physicists [MS, DV]; one finds mathematical treatments in [BE, FFRS].

Example 3.11. Let \mathcal{C} and \mathcal{D} be non-degenerate braided fusion categories. Recall from Section 2.4 that $\mathcal{C} \boxtimes \mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$ if and only if $\mathcal{C} \boxtimes \mathcal{D}$ contains a Lagrangian algebra $I(\mathbf{1})$ (here $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ is the right adjoint of the forgetful functor). Thus Theorem 3.6 implies that $\mathcal{D}^{\text{rev}} \simeq \mathcal{C}_A^0$ for some connected étale algebra $A \in \mathcal{C}$ if and only if there exists a braided equivalence $\mathcal{C} \boxtimes \mathcal{D} \simeq \mathcal{Z}(\mathcal{A})$ such that the forgetful functor $\mathcal{D} \rightarrow \mathcal{A}$ is injective. Moreover, in this case $\mathcal{C} \cap I(\mathbf{1}) = A$.

3.4. Subcategories of $\mathcal{Z}(\mathcal{A})$. Let \mathcal{A} be a fusion category and let $\mathcal{Z}(\mathcal{A})$ be its Drinfeld center with forgetful functor $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. Let $\mathcal{C} \subset \mathcal{Z}(\mathcal{A})$ be a fusion subcategory and let $\mathcal{C}' \subset \mathcal{Z}(\mathcal{A})$ be its Müger centralizer in $\mathcal{Z}(\mathcal{A})$.

Theorem 3.12. *The forgetful functor $\mathcal{C} \rightarrow \mathcal{A}$ is injective if and only if the forgetful functor $\mathcal{C}' \rightarrow \mathcal{A}$ is surjective.*

The proof of Theorem 3.12 is given in Section 3.5.

Remark 3.13. Let \mathcal{C}_1 and \mathcal{D} be non-degenerate braided fusion categories and let $A \in \mathcal{C}_1 \boxtimes \mathcal{D}$ be a connected étale algebra such that $A \cap \mathcal{C}_1 = \mathbf{1}$. Let $\mathcal{C}_2^{\text{rev}} = (\mathcal{C}_1 \boxtimes \mathcal{D})_A^0$. In view of Example 3.11 the conditions above are equivalent to the existence of a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{D} \boxtimes \mathcal{C}_2 \simeq \mathcal{Z}(\mathcal{A})$ such that

- (1) Forgetful functor $\mathcal{C}_1 \rightarrow \mathcal{A}$ is injective and
- (2) Forgetful functor $\mathcal{C}_1 \boxtimes \mathcal{D} \rightarrow \mathcal{A}$ is surjective.

Since $\mathcal{C}'_1 = \mathcal{D} \boxtimes \mathcal{C}_2$ and $(\mathcal{C}_1 \boxtimes \mathcal{D})' = \mathcal{C}_2$, Theorem 3.12 implies that the conditions above are equivalent to

- (1') Forgetful functor $\mathcal{D} \boxtimes \mathcal{C}_2 \rightarrow \mathcal{A}$ is surjective and
- (2') Forgetful functor $\mathcal{C}_2 \rightarrow \mathcal{A}$ is injective.

In view of Example 3.11 these conditions are equivalent to the existence of an étale algebra $B \in \mathcal{D} \boxtimes \mathcal{C}_2$ such that $B \cap \mathcal{C}_2 = \mathbf{1}$ and $(\mathcal{D} \boxtimes \mathcal{C}_2)_B^0 \simeq \mathcal{C}_1^{\text{rev}}$. Thus we proved the following result:

Theorem 3.14. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ be non-degenerate braided fusion categories. The existence of étale algebra $A \in \mathcal{C}_1 \boxtimes \mathcal{D}$ such that $A \cap \mathcal{C}_1 = \mathbf{1}$ and $(\mathcal{C}_1 \boxtimes \mathcal{D})_A^0 = \mathcal{C}_2^{\text{rev}}$ is equivalent to the existence of étale algebra $B \in \mathcal{D} \boxtimes \mathcal{C}_2$ such that $B \cap \mathcal{C}_2 = \mathbf{1}$ and $(\mathcal{D} \boxtimes \mathcal{C}_2)_B^0 \simeq \mathcal{C}_1^{\text{rev}}$. \square*

This Theorem is a special case¹ of [FFRS, Theorem 7.20]; see *loc. cit.* for the explanation of its significance for the Rational Conformal Field Theory.

¹In [FFRS, Theorem 7.20] the assumption on non-degeneracy of the categories \mathcal{C}_1 and \mathcal{C}_2 (but not $\mathcal{D}!$) is dropped.

3.5. Proof of Theorem 3.12. Let $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ be the right adjoint functor of the forgetful functor $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. Then $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{A})$ is an étale algebra as in Section 2.4 and the functor I naturally upgrades to a tensor equivalence of \mathcal{A} and the category $\mathcal{Z}(\mathcal{A})_{I(\mathbf{1})}$ of right $I(\mathbf{1})$ –modules, see [DMNO, Proposition 4.4]. For any étale subalgebra $B \subset I(\mathbf{1})$ let $\mathcal{A}(B) \subset \mathcal{A}$ consist of objects $X \in \mathcal{A}$ such that $I(X) \in \mathcal{Z}(\mathcal{A})_{I(\mathbf{1})}$ is a dyslectic B –module. Then it is proved in [DMNO, Theorem 4.10] that the assignment $B \mapsto \mathcal{A}(B)$ is an anti-isomorphism of the lattices of étale subalgebras of $I(\mathbf{1})$ and of fusion subcategories of \mathcal{A} . In addition we have

$$(15) \quad \text{FPdim}(\mathcal{A}(B)) = \frac{\text{FPdim}(\mathcal{A})}{\text{FPdim}(B)}.$$

Recall from Lemma 3.2 that for a fusion subcategory $\mathcal{C} \subset \mathcal{Z}(\mathcal{A})$, the subalgebra $\mathcal{C} \cap I(\mathbf{1}) \subset I(\mathbf{1})$ is étale.

Theorem 3.15. *Let $\mathcal{C} \subset \mathcal{Z}(\mathcal{A})$ be a fusion subcategory. Then $\mathcal{A}(\mathcal{C} \cap I(\mathbf{1}))$ is precisely the image $F(\mathcal{C}')$ of \mathcal{C}' in \mathcal{A} under the forgetful functor.*

Proof. It is clear that the image of \mathcal{C}' in $\mathcal{A} \simeq \mathcal{Z}(\mathcal{A})_{I(\mathbf{1})}$ consists of modules which are dyslectic when restricted to $\mathcal{C} \cap I(\mathbf{1})$, that is $F(\mathcal{C}') \subset \mathcal{A}(\mathcal{C} \cap I(\mathbf{1}))$. Hence

$$(16) \quad \text{FPdim}(F(\mathcal{C}')) \leq \text{FPdim}(\mathcal{A}(\mathcal{C} \cap I(\mathbf{1}))),$$

with equality if and only if $F(\mathcal{C}') = \mathcal{A}(\mathcal{C} \cap I(\mathbf{1}))$. Using (15) and [DMNO, Lemma 3.11] we see that (16) is equivalent to

$$(17) \quad \frac{\text{FPdim}(\mathcal{C}')}{\text{FPdim}(\mathcal{C}' \cap I(\mathbf{1}))} \leq \frac{\text{FPdim}(\mathcal{A})}{\text{FPdim}(\mathcal{C} \cap I(\mathbf{1}))}.$$

Interchanging \mathcal{C} and \mathcal{C}' we get

$$(18) \quad \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{C} \cap I(\mathbf{1}))} \leq \frac{\text{FPdim}(\mathcal{A})}{\text{FPdim}(\mathcal{C}' \cap I(\mathbf{1}))}.$$

Multiplying (17) and (18) and canceling the denominators we get

$$\text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{C}') \leq \text{FPdim}(\mathcal{A})^2.$$

However it follows from (5) and (6) that $\text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{C}') = \text{FPdim}(\mathcal{Z}(\mathcal{A})) = \text{FPdim}(\mathcal{A})^2$. Thus we had an equality in (16) and Theorem is proved. \square

Proof of Theorem 3.12. The injectivity of the forgetful functor $F : \mathcal{C} \rightarrow \mathcal{A}$ is equivalent to $\mathcal{C} \cap I(\mathbf{1}) = \mathbf{1}$. Since $B \mapsto \mathcal{A}(B)$ is an anti-isomorphism of lattices, the condition $\mathcal{A}(B) = \mathcal{A}$ is equivalent to $B = \mathbf{1}$. Thus Theorem 3.12 follows from Theorem 3.15. \square

4. TENSOR PRODUCT DECOMPOSITION OF A SLIGHTLY DEGENERATE BRAIDED FUSION CATEGORY

4.1. Non-degenerate braided fusion categories over a symmetric category.

Definition 4.1. Let \mathcal{E} be a symmetric fusion category and let \mathcal{C} be a braided fusion category over \mathcal{E} . We say that \mathcal{C} is *non-degenerate over \mathcal{E}* if $\mathcal{E} = \mathcal{C}'$, i.e., if \mathcal{E} coincides with the symmetric center of \mathcal{C} .

Remark 4.2. In this terminology a slightly degenerate braided fusion category is a non-degenerate braided fusion category over $s\text{Vec}$.

The following is a generalization of the decomposition theorem ([Mu2], [DGNO1, Theorem 3.13]) for non-degenerate braided fusion categories to the case of categories over \mathcal{E} .

Proposition 4.3. *Let \mathcal{C} be a non-degenerate braided fusion category over \mathcal{E} and let $\mathcal{D} \subset \mathcal{C}$ be its \mathcal{E} -subcategory. Suppose that \mathcal{D} is non-degenerate over \mathcal{E} . Then the centralizer \mathcal{D}' of \mathcal{D} in \mathcal{C} is also non-degenerate over \mathcal{E} and there is a braided tensor equivalence over \mathcal{E} :*

$$(19) \quad \mathcal{C} \cong \mathcal{D} \boxtimes_{\mathcal{E}} \mathcal{D}'.$$

Proof. Since \mathcal{D} is non-degenerate over \mathcal{E} , we have $\mathcal{D} \cap \mathcal{D}' \cong \mathcal{E}$. By (4) we have $\mathcal{D}'' = \mathcal{D} \vee \mathcal{C}' = \mathcal{D}$. Hence, $\mathcal{D}' \cap \mathcal{D}'' = \mathcal{D}' \cap \mathcal{D}$ and \mathcal{D}' is slightly degenerate.

To establish equivalence (19) first note that $\mathcal{D} \vee \mathcal{D}' = \mathcal{C}$. Indeed, $\mathcal{D} \vee \mathcal{D}'$ is a fusion subcategory of \mathcal{C} and

$$\text{FPdim}(\mathcal{D} \vee \mathcal{D}') = \frac{\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{D}')}{\text{FPdim}(\mathcal{E})} = \text{FPdim}(\mathcal{C})$$

by (5) and [DGNO1, Lemma 3.38]. The tensor multiplication defines a surjective braided tensor functor

$$\otimes : \mathcal{D} \boxtimes \mathcal{D}' \rightarrow \mathcal{D} \vee \mathcal{D}' = \mathcal{C}.$$

Let $R : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \mathcal{D}'$ be the right adjoint of \otimes . Since $R(\mathcal{E}) \subset \mathcal{E} \boxtimes \mathcal{E}$ we conclude that $\mathcal{D} \vee \mathcal{D}' \cong (\mathcal{D} \boxtimes \mathcal{D}')_{R(\mathbf{1})} \cong \mathcal{D} \boxtimes_{\mathcal{E}} \mathcal{D}'$, see Section 2.4 and Remark 2.10. \square

Let \mathcal{E} be a symmetric fusion category and let \mathcal{A} be a fusion category over \mathcal{E} such that the composition of $\mathcal{E} \rightarrow \mathcal{Z}(\mathcal{A})$ and the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ is fully faithful. We will denote by $\mathcal{Z}(\mathcal{A}, \mathcal{E})$ the centralizer of \mathcal{E} in $\mathcal{Z}(\mathcal{A})$. Observe that $\mathcal{Z}(\mathcal{A}, \mathcal{E})$ is a non-degenerate braided fusion \mathcal{E} -category. Let $F_{\mathcal{E}} : \mathcal{Z}(\mathcal{A}, \mathcal{E}) \rightarrow \mathcal{A}$ denote the forgetful functor. By Theorem 3.12 the functor $F_{\mathcal{E}}$ is surjective.

Corollary 4.4. *Let \mathcal{C} be a non-degenerate braided fusion category over \mathcal{E} . Then there is a braided tensor equivalence over \mathcal{E}*

$$(20) \quad \mathcal{Z}(\mathcal{C}, \mathcal{E}) \cong \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{C}^{\text{rev}}.$$

Proof. Let us view \mathcal{C} and \mathcal{C}^{rev} as fusion subcategories of $\mathcal{Z}(\mathcal{C})$. Clearly, they are centralizers of each other. By definition, $\mathcal{Z}(\mathcal{C}, \mathcal{E})$ is the centralizer of \mathcal{E} in $\mathcal{Z}(\mathcal{C})$, therefore, using [Mu2, Lemma 2.8] and (4) we get

$$\mathcal{Z}(\mathcal{C}, \mathcal{E}) = (\mathcal{C} \cap \mathcal{C}^{\text{rev}})' = \mathcal{C} \vee \mathcal{C}^{\text{rev}},$$

and the result follows from Proposition 4.3. \square

Let A be a connected étale algebra in $\mathcal{Z}(\mathcal{A}, \mathcal{E})$. The category $\text{Rep}_{\mathcal{A}}(A)$ of $F_{\mathcal{E}}(A)$ -modules in \mathcal{A} has a canonical structure of a *multi-fusion* category over \mathcal{E} (i.e., its unit object is not necessarily simple) with the embedding $\mathcal{E} \hookrightarrow \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$ given by the free module functor, i.e., $X \mapsto X \otimes F_{\mathcal{E}}(A)$ for all $X \in \mathcal{E}$.

Proposition 4.5. *There is a canonical braided tensor equivalence*

$$(21) \quad \mathcal{Z}(\mathcal{A}, \mathcal{E})_A^0 \cong \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A), \mathcal{E}).$$

Proof. There is a braided equivalence $\mathcal{Z}(\mathcal{A})_A^0 \cong \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$, see [Sch, Corollary 4.5] and [DMNO, Theorem 3.20]. It commutes with inclusions of \mathcal{E} and, hence, restricts to a braided equivalence (21). \square

Let A be a connected étale algebra in a non-degenerate braided fusion category \mathcal{C} over \mathcal{E} such that $A \cap \mathcal{E} = \mathbf{1}$. This algebra A can be considered as a connected étale algebra in \mathcal{C}^{rev} and in $\mathcal{Z}(\mathcal{C}, \mathcal{E})$ via the embedding

$$\mathcal{C}^{\text{rev}} = \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C}^{\text{rev}} \hookrightarrow \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{C}^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}, \mathcal{E}).$$

Under equivalence (20) we have

$$(22) \quad \mathcal{Z}(\mathcal{C}, \mathcal{E})_A \cong \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{C}_A^{\text{rev}} \quad \text{and} \quad \mathcal{Z}(\mathcal{C}, \mathcal{E})_A^0 \cong \mathcal{C} \boxtimes_{\mathcal{E}} (\mathcal{C}_A^0)^{\text{rev}}.$$

Corollary 4.6. *Let \mathcal{C} be a non-degenerate braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra such that $A \cap \mathcal{E} = \mathbf{1}$. There is a braided equivalence*

$$\mathcal{Z}(\mathcal{C}_A, \mathcal{E}) \cong \mathcal{C} \boxtimes_{\mathcal{E}} (\mathcal{C}_A^0)^{\text{rev}}.$$

In particular, the category \mathcal{C}_A^0 is non-degenerate over \mathcal{E} .

Proof. This is a direct consequence of equivalence (22) and Proposition 4.5. \square

Proposition 4.7. *Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories over \mathcal{E} . There is an equivalence of braided fusion categories over \mathcal{E}*

$$(23) \quad \mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{A}_2, \mathcal{E}) \cong \mathcal{Z}(\mathcal{A}_1 \boxtimes_{\mathcal{E}} \mathcal{A}_2, \mathcal{E}).$$

Proof. Start with $\mathcal{E} \boxtimes \mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2)$ and let \mathcal{F} denote the Tannakian subcategory of $\mathcal{E} \boxtimes \mathcal{E}$ such that $\mathcal{A}_1 \boxtimes_{\mathcal{E}} \mathcal{A}_2 = (\mathcal{A}_1 \boxtimes \mathcal{A}_2) \boxtimes_{\mathcal{F}} \text{Vec}$, see Remark 2.11.

Consider the centralizers of $\mathcal{E} \boxtimes \mathcal{E}$ and \mathcal{F} in $\mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2)$. We have embedding $\mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{E} \boxtimes \mathcal{E}) \subset \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{F})$.

By Proposition 2.12 braided fusion category $\mathcal{Z}(\mathcal{A}_1 \boxtimes_{\mathcal{E}} \mathcal{A}_2)$ is equivalent to the de-equivariantization $\mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{F}) \boxtimes_{\mathcal{F}} \text{Vec}$. The category $\mathcal{Z}(\mathcal{A}_1 \boxtimes_{\mathcal{E}} \mathcal{A}_2, \mathcal{E})$ therefore is identified with the centralizer of $(\mathcal{E} \boxtimes \mathcal{E}) \boxtimes_{\mathcal{F}} \text{Vec} \cong \mathcal{E}$ in $\mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{F}) \boxtimes_{\mathcal{F}} \text{Vec}$.

On the other hand, $\mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \boxtimes \mathcal{Z}(\mathcal{A}_2, \mathcal{E}) = \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{E} \boxtimes \mathcal{E})$ and, therefore, $\mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{A}_2, \mathcal{E})$ is identified with $\mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{E} \boxtimes \mathcal{E}) \boxtimes_{\mathcal{F}} \text{Vec}$.

Thus, existence of the braided equivalence (23) follows from Proposition 2.13. \square

4.2. Slightly degenerate braided fusion categories.

Definition 4.8. A braided s -category is a braided fusion category over $s\text{Vec}$.

In other words, a braided s -category is a braided fusion category \mathcal{C} equipped with a braided tensor functor $T : s\text{Vec} \rightarrow \mathcal{C}'$. Such a functor T is necessarily injective.

An s -subcategory of a braided s -category \mathcal{C} is a fusion subcategory $\mathcal{C}_1 \subset \mathcal{C}$ containing $T(s\text{Vec})$. Note that an s -subcategory of a braided s -category is a fusion category over $s\text{Vec}$.

Definition 4.9. A braided s -category \mathcal{C} is called *s-simple* if it is non-pointed and has no s -subcategories except $s\text{Vec}$ and \mathcal{C} .

Remark 4.10. An s -simple category is slightly degenerate (see Definition 2.1).

Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be braided s -categories. One defines their tensor product over $s\text{Vec}$ by iterating the construction of Section 2.5. The equivalence class of the resulting braided s -category does not depend on the order in which products are taken. By transitivity of de-equivariantization (see [DGNO1, Lemma 4.32]) there is a braided tensor equivalence

$$\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n \cong (\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n) \boxtimes_{\tilde{\mathcal{E}}} \text{Vec},$$

where $\tilde{\mathcal{E}}$ is the maximal Tannakian subcategory of $(s\text{Vec})^{\boxtimes n} \subset \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$.

Clearly, $\tilde{\mathcal{E}} \cong \text{Rep}(G)$ where $G \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$ is an elementary Abelian 2-group. There is canonical action of the group G on the tensor product $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n$ by braided tensor autoequivalences, see [DGNO1, Section 4].

Proposition 4.11. *Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be slightly degenerate braided fusion categories. The canonical action of G on $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n$ is identical on objects, i.e., $g(X) \cong X$ for all objects $X \in \mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n$ and all $g \in G$.*

Proof. Let $A = \text{Fun}(G)$ be the regular algebra in $\tilde{\mathcal{E}}$. Recall that the action of G on $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n = (\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n)_A$ is given by the action of G on A by translations. In particular, $g(X) \cong X$ for every free A -module $X = A \otimes Y$, $Y \in \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ and $g \in G$. Thus, it suffices to show that every simple A -module in $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n$ is free. Since the functor of taking free A -module

$$(\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n) \rightarrow (\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n)_A : Y \mapsto Y \otimes A$$

is surjective, the last property is equivalent to every free A -module being simple.

In our situation A is equal to the direct sum of objects $\delta_1 \boxtimes \cdots \boxtimes \delta_n$ in $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$, where each δ_i , $i = 1, \dots, n$, is isomorphic to $\mathbf{1}$ or δ and the number of i 's for which $\delta_i \cong \delta$ is even. Since $\delta \otimes Y \not\cong Y$ for any simple object Y of a slightly degenerate category (see Section 2.2) it follows that every free A -module in $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ is a direct sum of 2^{n-1} non-isomorphic simple objects in $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$. It is easy to see that every such A -module is necessarily simple. \square

Corollary 4.12. *Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be slightly degenerate braided fusion categories. Let $\tilde{\mathcal{E}}$ denote the maximal Tannakian subcategory of $(s\text{Vec})^{\boxtimes n} \hookrightarrow \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$. There is an isomorphism between the lattice of fusion subcategories of $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ containing $\tilde{\mathcal{E}}$ and the lattice of fusion subcategories of $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n$ given by*

$$\mathcal{D} \mapsto \mathcal{D} \boxtimes_{\tilde{\mathcal{E}}} \text{Vec}.$$

Proof. By [DGNO1, Proposition 4.30(i)] there is an isomorphism between the lattice of fusion subcategories of $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ containing $\tilde{\mathcal{E}}$ and the lattice of G -stable fusion subcategories of $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_n = (\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n) \boxtimes_{\tilde{\mathcal{E}}} \text{Vec}$ given by $\mathcal{D} \mapsto \mathcal{D} \boxtimes_{\tilde{\mathcal{E}}} \text{Vec}$. So the statement follows from Proposition 4.11. \square

4.3. Decomposition Theorem. The next Theorem is a special case of a result from [DGNO2]. We include the proof for the reader's convenience.

Theorem 4.13. *Let \mathcal{C} be a slightly degenerate braided fusion category. Suppose \mathcal{C} has no Tannakian subcategories other than Vec . Then*

- (1) *There exist s -simple subcategories $\mathcal{C}_1, \dots, \mathcal{C}_m \subset \mathcal{C}$ determined uniquely up to a permutation of indices such that*

$$(24) \quad \mathcal{C} \cong \mathcal{C}_{pt} \boxtimes_{s\text{Vec}} \mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m.$$

- (2) *Every fusion subcategory $\mathcal{D} \subset \mathcal{C}$ has the form*

$$\mathcal{D} = \mathcal{D}_{pt} \boxtimes_{s\text{Vec}} \mathcal{C}_{i_1} \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_{i_k}$$

for a subset $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$.

Proof. Since \mathcal{C} has no non-trivial Tannakian subcategories, we see that Vec and $\mathcal{C}' = \text{sVec}$ are the only symmetric subcategories of \mathcal{C} .

For any fusion subcategory $\mathcal{D} \subset \mathcal{C}$ the category $\mathcal{D} \cap \mathcal{D}'$ is symmetric. Therefore, \mathcal{D} is either non-degenerate (if $\text{sVec} \not\subset \mathcal{D}$) or slightly degenerate (if $\text{sVec} \subset \mathcal{D}$).

In particular, \mathcal{C}_{pt} is slightly degenerate. Therefore, \mathcal{C}'_{pt} is slightly degenerate and $\mathcal{C} \cong \mathcal{C}_{pt} \boxtimes_{\text{sVec}} \mathcal{C}'_{pt}$ by Proposition 4.3. So it suffices to prove the Theorem in the case when $\mathcal{C}_{pt} = \text{sVec}$.

If \mathcal{C} is s-simple then there is nothing to prove. Otherwise, let \mathcal{C}_1 be an s-simple subcategory of \mathcal{C} . Then $\mathcal{C} \cong \mathcal{C}_1 \boxtimes_{\text{sVec}} \mathcal{C}'_1$ by Proposition 4.3. If \mathcal{C}'_1 is not s-simple choose its s-simple subcategory \mathcal{C}_2 and apply Proposition 4.3 again. Continuing in this way we obtain decomposition (24).

Let us prove the uniqueness assertion. Let \mathcal{D} be an s-subcategory of \mathcal{C} . By Corollary 4.12 there is a fusion subcategory $\tilde{\mathcal{D}} \subset \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ containing $(\text{sVec})^{\boxtimes n}$ such that

$$\mathcal{D} \cong \tilde{\mathcal{D}} \boxtimes_{\mathcal{E}} \text{Vec}.$$

Let $\tilde{\mathcal{D}}_i \subset \mathcal{C}_i$ be the fusion subcategory consisting of all objects $X_i \in \mathcal{C}_i$ such that there is an object $X = X_1 \boxtimes \cdots \boxtimes X_i \boxtimes \cdots \boxtimes X_n \in \tilde{\mathcal{D}}$. Since \mathcal{C}_i is s-simple we have either $\tilde{\mathcal{D}}_i = \text{sVec}$ or $\tilde{\mathcal{D}}_i = \mathcal{C}_i$. Let us analyze the former case. Let $\mathcal{B}_i \subset \mathcal{C}_i$ be a fusion subcategory generated by objects contained in $X_i \otimes X_i^*$, where X_i is simple object in $\tilde{\mathcal{D}}_i$. Since

$X \otimes X^* = (X_1 \otimes X_1^*) \boxtimes \cdots \boxtimes (X_n \otimes X_n^*) \succ \mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1} \boxtimes (X_i \otimes X_i^*) \boxtimes \mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1}$, the category \mathcal{B}_i is contained in $\tilde{\mathcal{D}}$. We claim that $\mathcal{B}_i = \mathcal{C}_i$. Indeed, otherwise $\mathcal{B}_i = \text{sVec}$ since \mathcal{C}_i is s-simple. Hence, $\text{FPdim}(Y) = \sqrt{2}$ for every simple non-invertible object $Y \in \tilde{\mathcal{D}}_i$. Therefore, $Y \otimes Y = \mathbf{1} \oplus \delta$ and so Y generates an Ising subcategory $\mathcal{I} \subset \tilde{\mathcal{D}}_i$. But this is impossible since on the one hand \mathcal{I} is non-degenerate (see Section 2.2) and on the other hand $\tilde{\mathcal{D}}'_i = \text{sVec} \subset \mathcal{I} \cap \mathcal{I}'$. We conclude that either $\tilde{\mathcal{D}}_i = \text{sVec}$ or $\mathcal{C}_i \subset \mathcal{D}$. Therefore,

$$\mathcal{D} = (\text{sVec})^{\boxtimes n} \vee (\mathcal{C}_{i_1} \boxtimes_{\text{sVec}} \cdots \boxtimes_{\text{sVec}} \mathcal{C}_{i_k}),$$

where $\{i_1, \dots, i_k\}$ is a subset of $\{1, 2, \dots, n\}$. It follows that

$$\mathcal{D} \cong \mathcal{C}_{i_1} \boxtimes_{\text{sVec}} \cdots \boxtimes_{\text{sVec}} \mathcal{C}_{i_k}.$$

In particular, if \mathcal{D} is s-simple then $\mathcal{D} = \mathcal{C}_i$ for some i . This implies uniqueness of the tensor decomposition of \mathcal{C} . \square

5. THE WITT GROUP OF SLIGHTLY DEGENERATE BRAIDED FUSION CATEGORIES

5.1. The Witt group of non-degenerate braided fusion categories over \mathcal{E} .

Definition 5.1. Let \mathcal{C}_1 and \mathcal{C}_2 be non-degenerate braided fusion categories over a symmetric fusion category \mathcal{E} such that the corresponding braided tensor functors $\mathcal{E} \rightarrow \mathcal{C}_i$, $i = 1, 2$, are fully faithful. We will say that \mathcal{C}_1 and \mathcal{C}_2 are *Witt equivalent* if there exist fusion categories $\mathcal{A}_1, \mathcal{A}_2$ over \mathcal{E} and a braided equivalence over \mathcal{E}

$$\mathcal{C}_1 \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \cong \mathcal{C}_2 \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{A}_2, \mathcal{E}).$$

It follows from Proposition 4.7 that the Witt equivalence of non-degenerate braided fusion categories over \mathcal{E} is indeed an equivalence relation. We will denote the Witt equivalence class of a non-degenerate braided fusion \mathcal{E} -category \mathcal{C} by $[\mathcal{C}]$. The set of Witt equivalence classes of slightly degenerate braided fusion

categories will be denoted $\mathcal{W}(\mathcal{E})$. Clearly, $\mathcal{W}(\mathcal{E})$ is a commutative monoid with respect to the multiplication $\boxtimes_{\mathcal{E}}$. The unit of this monoid is $[\mathcal{E}]$.

Lemma 5.2. *The monoid $\mathcal{W}(\mathcal{E})$ is a group.*

Proof. This immediately follows from Corollary 4.4. \square

Proposition 5.3. *Let A be a connected étale algebra in a non-degenerate category \mathcal{C} over \mathcal{E} such that $A \cap \mathcal{E} = \mathbf{1}$. Then $[\mathcal{C}] = [\mathcal{C}_A^0]$ in $\mathcal{W}(\mathcal{E})$.*

Proof. The result follows from Corollary 4.6. \square

Definition 5.4. Let \mathcal{E} be a symmetric fusion category and let \mathcal{C} be a braided fusion category over \mathcal{E} . We say that \mathcal{C} is *completely \mathcal{E} -anisotropic* if every connected étale algebra in \mathcal{C} belongs to \mathcal{E} .

For $\mathcal{E} = \text{Vec}$, sVec the above notion coincides with the usual complete anisotropy, cf. Section 2.4.

Theorem 5.5. *Each equivalence class in $\mathcal{W}(\mathcal{E})$ contains a completely \mathcal{E} -anisotropic category category that is unique up to a braided equivalence.*

Proof. This is completely parallel to [DMNO, Theorem 5.13]. \square

Proposition 5.6. *Let \mathcal{C} be a non-degenerate braided fusion category over \mathcal{E} . Then $\mathcal{C} \in [\mathcal{E}]$ if and only if there exist a fusion category \mathcal{A} over \mathcal{E} and an equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{A}, \mathcal{E})$ of braided fusion categories over \mathcal{E} .*

Proof. This is completely parallel to [DMNO, Proposition 5.8]. By definition, $\mathcal{C} \in [\mathcal{E}]$ if and only if

$$\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \cong \mathcal{Z}(\mathcal{A}_2, \mathcal{E}).$$

Let $A_1 \in \mathcal{Z}(\mathcal{A}_1, \mathcal{E})$ be the connected étale algebra corresponding to the forgetful central functor $\mathcal{Z}(\mathcal{A}_1, \mathcal{E}) \rightarrow \mathcal{A}_1$. We have

$$\mathcal{C} \cong \mathcal{Z}(\mathcal{A}_2, \mathcal{E})_{A_1}^0 \cong \mathcal{Z}(\text{Rep}_{\mathcal{A}_2}(A_1), \mathcal{E}),$$

where the last equivalence is by Proposition 4.5. \square

We have $\mathcal{W}(\text{Vec}) = \mathcal{W}$, the Witt group of non-degenerate braided fusion categories introduced in [DMNO], see Section 2.7.

Let us denote $s\mathcal{W} := \mathcal{W}(\text{sVec})$.

Definition 5.7. The group $s\mathcal{W}$ is called the *Witt group of slightly degenerate braided fusion categories*.

The group $s\mathcal{W}$ is the main object of our analysis in the remainder of this paper.

Remark 5.8. The base change along $\mathcal{E} \rightarrow \mathcal{F}$ defines a group homomorphism $\mathcal{W}(\mathcal{E}) \rightarrow \mathcal{W}(\mathcal{F})$. Since by Deligne's theorem [De] any symmetric fusion category \mathcal{E} has a surjective braided tensor functor either into Vec or to sVec (i.e., \mathcal{E} is either Tannakian or super-Tannakian) it follows that each $\mathcal{W}(\mathcal{E})$ has a canonical homomorphism either to \mathcal{W} or to $s\mathcal{W}$.

5.2. Relations in $s\mathcal{W}$. Let \mathcal{C} be a completely anisotropic slightly degenerate braided fusion category and let

$$(25) \quad \mathcal{C} \cong \mathcal{C}_{pt} \boxtimes_{s\text{Vec}} \mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m,$$

be its decomposition into tensor product of s -simple subcategories from Theorem 4.13. Then categories $\mathcal{C}_{pt}, \mathcal{C}_1, \dots, \mathcal{C}_m$ are completely anisotropic.

Next Theorem extends [DMNO, Theorem 5.19] to slightly degenerate categories. It describes all central functors on tensor products of completely anisotropic s -simple braided fusion categories (and, hence, all connected étale algebras in such products).

Theorem 5.9. *Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be completely anisotropic s -simple braided fusion categories. Let*

$$F : \mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m \rightarrow \mathcal{A}$$

be a surjective central tensor functor.

(1) *There is a subset $J \subset \{1, \dots, m\}$ such that*

$$\mathcal{A} \cong \bigboxtimes_{\substack{s\text{Vec} \\ j \in J}} \mathcal{C}_j.$$

(2) *There is a surjective map $f : \{1, \dots, m\} \rightarrow J$ such that $|f^{-1}(j)| \leq 2$ for all $j \in J$ and*

$$F = \bigboxtimes_{\substack{s\text{Vec} \\ j \in J}} F_j, \quad \text{where} \quad F_j : \bigboxtimes_{\substack{s\text{Vec} \\ i \in f^{-1}(j)}} \mathcal{C}_i \rightarrow \mathcal{C}_j \quad \text{is the restriction of } F.$$

(3) *If $f^{-1}(j) = \{i\}$ then $F_j : \mathcal{C}_i \xrightarrow{\sim} \mathcal{C}_j$ is tensor equivalence. If $f^{-1}(j) = \{i, i'\}$ then there is a braided tensor equivalence $\mathcal{C}_{i'} \cong \mathcal{C}_i^{\text{rev}}$ and F_j is the forgetful tensor functor*

$$(26) \quad \mathcal{C}_i \boxtimes_{s\text{Vec}} \mathcal{C}_{i'} \cong \mathcal{C}_j \boxtimes_{s\text{Vec}} \mathcal{C}_j^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}_j, s\text{Vec}) \xrightarrow{\text{Forget}} \mathcal{C}_j \cong \mathcal{C}_i.$$

Proof. Let $\mathcal{A}_j \subset \mathcal{A}$ denote the image of \mathcal{C}_j in \mathcal{A} , $j = 1, \dots, m$. Since \mathcal{C}_j is completely anisotropic, we have $\mathcal{A}_j \cong \mathcal{C}_j$. Clearly, $\mathcal{A} = \bigvee_{j=1}^m \mathcal{A}_j$.

The set J is formed by induction as follows. Set $J_1 = \{1\}$. Suppose for some $k < m$ a subset J_k is chosen in such way that

$$\bigvee_{j=1}^k \mathcal{A}_j \cong \bigboxtimes_{\substack{s\text{Vec} \\ j \in J_k}} \mathcal{C}_j.$$

Since $\mathcal{A}_{k+1} \cong \mathcal{C}_{k+1}$ is s -simple we have either $\mathcal{A}_{k+1} \subset \bigvee_{j=1}^k \mathcal{A}_j$ or $\mathcal{A}_{k+1} \cap (\bigvee_{j=1}^k \mathcal{A}_j) = s\text{Vec}$. In the former case set $J_{k+1} = J_k$. Clearly, we have

$$\bigvee_{j=1}^{k+1} \mathcal{A}_j \cong \bigboxtimes_{\substack{s\text{Vec} \\ j \in J_{k+1}}} \mathcal{C}_j.$$

In the latter case $\bigvee_{j=1}^{k+1} \mathcal{A}_j$ is generated by two fusion subcategories, $\bigvee_{j=1}^k \mathcal{A}_j$ and \mathcal{A}_{k+1} , whose intersection is $s\text{Vec}$. The composition of F with the tensor product of \mathcal{C} gives rise to a surjective tensor functor

$$(27) \quad \left(\bigboxtimes_{\substack{s\text{Vec} \\ j \in J_k}} \mathcal{C}_j \right) \boxtimes_{s\text{Vec}} \mathcal{C}_{k+1} \rightarrow \bigvee_{j=1}^{k+1} \mathcal{A}_j.$$

It follows from [DGNO1, Lemma 3.38] that both sides of (27) have equal Frobenius-Perron dimension. Therefore, (27) is an equivalence and $J := J_n$ is the required subset. This proves part (1).

The functor F factors as

$$\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m \xrightarrow{F'} \mathcal{Z}(\mathcal{A}) \cong \bigotimes_{\substack{s\text{Vec} \\ j \in J}} (\mathcal{C}_j \boxtimes_{s\text{Vec}} \mathcal{C}_j^{\text{rev}}) \rightarrow \mathcal{A} \cong \bigotimes_{\substack{s\text{Vec} \\ j \in J}} \mathcal{C}_j.$$

Since $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m$ is slightly degenerate, the braided tensor functor F' is injective. Define an embedding $\{1, \dots, m\} \hookrightarrow J \sqcup J$ by $f(i) = j \in J \sqcup \emptyset$ if $F'(\mathcal{C}_i) = \mathcal{C}_j$ and $f(i) = j \in \emptyset \sqcup J$ if $F'(\mathcal{C}_i) = \mathcal{C}_j^{\text{rev}}$. This embedding is well defined by Theorem 4.13. Let

$$f : \{1, \dots, m\} \hookrightarrow J \sqcup J \rightarrow J$$

to be the composition of the above embedding and the diagonal map. Since F is surjective, so is f . Since $\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m$ is slightly degenerate the braided tensor functor F' is injective. Therefore, $f^{-1}(j)$ contains at most 2 elements for every $j \in J$ and F decomposes as a product of central tensor functors

$$F_j : \bigotimes_{\substack{s\text{Vec} \\ i \in f^{-1}(j)}} \mathcal{C}_i \rightarrow \mathcal{C}_j.$$

This proves part (2). Part (3) is an immediate consequence of the above definition of f . \square

Corollary 5.10. *Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be completely anisotropic s -simple braided categories. Suppose that there exists a fusion category \mathcal{A} over $s\text{Vec}$ such that*

$$\mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_m \cong \mathcal{Z}(\mathcal{A}, s\text{Vec}).$$

Then there exists a fixed point free involution a of the set $\{1, \dots, m\}$ such that $\mathcal{C}_i \cong \mathcal{C}_{a(i)}^{\text{rev}}$ for all $i = 1, \dots, m$.

Proof. Let us use the notation of Theorem 5.9. In this case $\{1, \dots, m\} = J \sqcup J$ and the map $f : \{1, \dots, m\} \rightarrow J$ is two-to-one. Hence, it gives rise to a fixed point free involution of $\{1, \dots, m\}$. \square

Remark 5.11. Corollary 5.10 means that there are no non-trivial relations between the Witt classes of completely anisotropic s -simple braided categories in $s\mathcal{W}$ except possibly relations of the form $[\mathcal{C}] = [\mathcal{C}]^{-1}$.

Corollary 5.12. *Let \mathcal{C} be a completely anisotropic slightly degenerate braided fusion category. Suppose that $\mathcal{C}_{pt} = s\text{Vec}$. If $[\mathcal{C}] \neq [s\text{Vec}]$ then the order of $[\mathcal{C}]$ in $s\mathcal{W}$ is either 2 or ∞ .*

Proof. By Theorem 4.13 we have $\mathcal{C} \cong \mathcal{C}_1 \boxtimes_{s\text{Vec}} \cdots \boxtimes_{s\text{Vec}} \mathcal{C}_k$, where $\mathcal{C}_1, \dots, \mathcal{C}_k$ are completely anisotropic s -simple categories.

By Proposition 5.6 the order of $[\mathcal{C}]$ is finite if and only if $\mathcal{C}^{\boxtimes n} \cong \mathcal{Z}(\mathcal{A}, s\text{Vec})$ for some fusion category \mathcal{A} over $s\text{Vec}$. It follows from Corollary 5.10 that there is an involution a of $\{1, \dots, k\}$ (not necessarily fixed point free) such that $\mathcal{C}_i \cong \mathcal{C}_{a(i)}^{\text{rev}}$ for each $i = 1, \dots, k$. Thus, after cancellations $[\mathcal{C}]$ is equal to the product of classes $[\mathcal{C}_i]$ with $\mathcal{C}_i \cong \mathcal{C}_i^{\text{rev}}$, i.e., of elements of order 2. \square

5.3. A canonical homomorphism $\mathcal{W} \rightarrow s\mathcal{W}$. Let \mathcal{W} be the Witt group of non-degenerate braided fusion categories, see Section 2.7 and [DMNO]. Define a map

$$(28) \quad S : \mathcal{W} \rightarrow s\mathcal{W} : [\mathcal{C}] \mapsto [\mathcal{C} \boxtimes \text{sVec}].$$

Proposition 5.13. *The map (28) is a well defined homomorphism.*

Proof. It suffices to check that (a) S maps the trivial class in \mathcal{W} to the trivial class in $s\mathcal{W}$ and that (b) S is multiplicative. Let \mathcal{C} be a non-degenerate braided fusion category such that $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$ for some fusion category \mathcal{A} . Let us view sVec as a symmetric subcategory of $\mathcal{Z}(\text{Vec}_{\mathbb{Z}/2\mathbb{Z}})$. Then $\mathcal{C} \boxtimes \text{sVec}$ is the centralizer of $\text{Vec} \boxtimes \text{sVec}$ in $\mathcal{C} \boxtimes \mathcal{Z}(\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}) \cong \mathcal{Z}(\mathcal{A} \boxtimes \text{Vec}_{\mathbb{Z}/2\mathbb{Z}})$, i.e., the class of $S([\mathcal{C}]) = [\mathcal{C} \boxtimes \text{sVec}]$ in $s\mathcal{W}$ is trivial. This proves (a). The verification of (b) is straightforward:

$$S([\mathcal{C}_1])S([\mathcal{C}_2]) = [(\mathcal{C}_1 \boxtimes \text{sVec}) \underset{\text{sVec}}{\boxtimes} (\mathcal{C}_2 \boxtimes \text{sVec})] = [\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \text{sVec}] = S([\mathcal{C}_1][\mathcal{C}_2]),$$

for all non-degenerate braided fusion categories \mathcal{C}_1 and \mathcal{C}_2 . \square

Recall from Section 2.7 that \mathcal{W} contains a cyclic subgroup \mathcal{W}_{Ising} of order 16 generated by the Witt classes of Ising braided fusion categories.

Proposition 5.14. *The kernel of homomorphism (28) is \mathcal{W}_{Ising} .*

Proof. Let \mathcal{C} be a non-degenerate braided fusion category such that

$$\mathcal{C} \boxtimes \text{sVec} \cong \mathcal{Z}(\mathcal{A}, \text{sVec})$$

for some fusion category \mathcal{A} over sVec . Then $\mathcal{Z}(\mathcal{A}) \cong \mathcal{C} \boxtimes \mathcal{C}_1$ where \mathcal{C}_1 is a non-degenerate braided fusion category of the Frobenius-Perron dimension 4 containing sVec . It is easy to see that categories with the above property are precisely Ising braided categories and pointed braided categories $\mathcal{C}(A, q)$ associated with metric groups (A, q) of order 4 such that there exists $u \in A$ with $q(u) = -1$. Since $[\mathcal{C}] = [\mathcal{C}_1]^{-1}$ in \mathcal{W} we see that the kernel of S is \mathcal{W}_{Ising} (see Section 2.7). \square

Question 5.15. At the moment of writing we do not know whether the homomorphism (28) is surjective. More generally, can one always embed a slightly degenerate braided fusion category \mathcal{C} into a non-degenerate braided fusion category \mathcal{D} such that $\text{FPdim}(\mathcal{D}) = 2\text{FPdim}(\mathcal{C})$?

5.4. The structure of $s\mathcal{W}$. Let $s\mathcal{W}_{pt}$ denote the subgroup of $s\mathcal{W}$ generated by the Witt classes of slightly degenerate pointed braided fusion categories.

Proposition 5.16. *We have*

$$s\mathcal{W}_{pt} = \bigoplus_{p \text{ is prime}} s\mathcal{W}_{pt}(p),$$

where $s\mathcal{W}_{pt}(p)$ is the group generated by classes of slightly degenerate pre-metric p -groups:

$$s\mathcal{W}_{pt}(p) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p \equiv 1 \pmod{4}, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. By [DGNO1, Corollary A.19] $s\mathcal{W}_{pt}$ is a surjective image of the restriction of homomorphism (28) on the subgroup

$$\mathcal{W}_{pt} = \bigoplus_{p \text{ is prime}} \mathcal{W}_{pt}(p) \subset \mathcal{W}.$$

The intersection of \mathcal{W}_{pt} with $\text{Ker}(S) = \mathcal{W}_{Ising}$ is contained in $\mathcal{W}_{pt}(2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, whence $s\mathcal{W}_{pt}(2) \cong \mathbb{Z}/2\mathbb{Z}$ and $s\mathcal{W}_{pt}(p) \cong \mathcal{W}_{pt}(p)$ for $p > 2$. \square

Corollary 5.17. *The group $s\mathcal{W}$ is 2-primary, i.e., it has no non-trivial elements of odd order.*

Proof. Let \mathcal{C} be a completely anisotropic slightly degenerate braided fusion category. Then $\mathcal{C} \cong \mathcal{C}_{pt} \boxtimes_{s\text{Vec}} \mathcal{C}'_{pt}$ and the category \mathcal{C}'_{pt} satisfies conditions of Corollary 5.12. So if the order of $[\mathcal{C}]$ in $s\mathcal{W}$ is finite then the order of $[\mathcal{C}'_{pt}]$ in $s\mathcal{W}$ is at most 2. We have

$$[\mathcal{C}] = [\mathcal{C}_{pt}][\mathcal{C}'_{pt}].$$

By Proposition 5.16 the order of $[\mathcal{C}_{pt}]$ in $s\mathcal{W}$ is even. Therefore, the order of $[\mathcal{C}]$ is even. \square

Let $s\mathcal{W}_2$ (respectively, $s\mathcal{W}_\infty$) denote the subgroups of $s\mathcal{W}$ generated by the Witt classes of completely anisotropic s -simple braided fusion categories of the the Witt order 2 (respectively, of infinite Witt order).

Proposition 5.18. *We have*

$$(29) \quad s\mathcal{W} = s\mathcal{W}_{pt} \bigoplus s\mathcal{W}_2 \bigoplus s\mathcal{W}_\infty.$$

The subgroup $s\mathcal{W}_2$ is an elementary Abelian 2-group and the subgroup $s\mathcal{W}_\infty$ is a free Abelian group of countable rank.

Proof. That $s\mathcal{W} = s\mathcal{W}_{pt} + s\mathcal{W}_2 + s\mathcal{W}_\infty$ is a consequence of Theorem 4.13 and Corollary 5.12. By Corollary 5.10 the only non-trivial relations in $s\mathcal{W}$ involving the classes of s -simple categories are of the form $2[\mathcal{C}] = 0$ for $[\mathcal{C}] \in s\mathcal{W}_2$. This proves that the sum is direct and that $s\mathcal{W}_\infty$ is free. By [DMNO, Section 6.4] the Witt group \mathcal{W} contains a free Abelian subgroup of countable rank. The homomorphism S embeds this subgroup into $s\mathcal{W}_\infty$, so the latter has a countable rank as well. \square

Corollary 5.19. *The group \mathcal{W} is 2-primary. The maximal finite order of an element of \mathcal{W} is 32.*

Proof. Let \mathcal{C} be a completely anisotropic non-degenerate braided fusion category. If the order of $[\mathcal{C}]$ in \mathcal{W} is odd then $s\mathcal{W}$ contains a non-trivial element of odd order, which is impossible by Corollary 5.17. Thus, \mathcal{W} is 2-primary.

Suppose that the order of $[\mathcal{C}]$ in \mathcal{W} is finite and ≥ 64 . We have $\mathcal{C} \cong \mathcal{P} \boxtimes \mathcal{C}_1$, where \mathcal{P} is a pointed non-degenerate braided fusion category and \mathcal{C}_1 is non-degenerate and such that $(\mathcal{C}_1)_{pt} = \text{Vec}$ or $(\mathcal{C}_1)_{pt} = s\text{Vec}$. Then the order of $[\mathcal{C}_1]$ in \mathcal{W} is ≥ 64 and the order of $S([\mathcal{C}_1]) = [\mathcal{C}_1 \boxtimes s\text{Vec}]$ in $s\mathcal{W}$ is $\geq \frac{64}{16} = 4$. By Theorem 5.18 we have $[\mathcal{C}_1] \in s\mathcal{W}_2$ so the order of $[\mathcal{C}_1]$ is at most 2, a contradiction.

Note that \mathcal{W} does contain an element of order 32, namely the Witt class of $C(sl(2), 6)$ (see [DMNO, Section 6.4]). \square

Remark 5.20. The subgroup $s\mathcal{W}_2$ is non-zero. Indeed, it contains non-zero elements $S([\mathcal{C}(so(2n+1), 2n+1)])$, $n \geq 1$. From the existence of conformal embeddings $so(m)_m \times so(m)_m \subset so(m^2)_1$ we see that the Witt classes $[\mathcal{C}(so(2n+1), 2n+1)]$ are square roots of the classes of Ising categories. Hence, they all have order 32 in the Witt group \mathcal{W} . Therefore, $S([\mathcal{C}(so(2n+1), 2n+1)]) \in s\mathcal{W}_2$ by Corollary 5.12.

We conjecture that the classes $S([\mathcal{C}(so(2n+1), 2n+1)])$, $n \geq 1$, are pairwise Witt non-equivalent in $s\mathcal{W}$, so that $s\mathcal{W}_2$ has an infinite order.

5.5. The subgroup of \mathcal{W} generated by classes $[\mathcal{C}(sl(2), k)]$, $k \geq 1$. We use the notations and observations made in [DMNO, Section 6.4]. For the basic facts about categories $\mathcal{C}(sl(2), k)$ see section 2.8.

It follows from the fusion rules of $\mathcal{C}(sl(2), k)$ that the object $[k]$ is invertible of order 2: $[k] \otimes [k] = [0]$ and it generates $\mathcal{C}(sl(2), k)_{pt}$ (i.e., it is the only non-trivial invertible object). Its tensor product with simple objects has the form $[k] \otimes [i] = [k-i]$. The centralizer $\mathcal{C}(sl(2), k)_+$ of $[k]$ in $\mathcal{C}(sl(2), k)$ is the full subcategory with simple objects $[2s]$, $s = 0, \dots, 2l+1$ of “even spin”. The braiding of $[k]$ with itself is well known:

$$c_{[k],[k]} = \theta_{[k]} = e^{2\pi i \frac{k}{4}} \text{id}_{[0]}.$$

Consider three different cases of the level : $k = 2l+1$, $k = 4l$, and $k = 4l+2$.

For $k = 2l+1$ the pointed part $\mathcal{C}(sl(2), k)_{pt}$ is non-degenerate. By Müger’s decomposition theorem [Mu2]

$$\mathcal{C}(sl(2), k) \simeq \mathcal{C}(sl(2), k)_{pt} \boxtimes \mathcal{C}(sl(2), k)_+.$$

It can be seen by looking at the fusion rules that $\mathcal{C}(sl(2), k)_+$ is simple. It is also completely anisotropic. Note that $\mathcal{C}(sl(2), 1)_{pt} = \mathcal{C}(sl(2), 1)$ and its class has order 8 in the Witt group \mathcal{W} :

$$(30) \quad [\mathcal{C}(sl(2), 1)]^8 = 1.$$

Moreover we have the following relation between Witt classes

$$[\mathcal{C}(sl(2), 2l+1)_{pt}] = [\mathcal{C}(sl(2), 1)]^{(-1)^l}.$$

For $k = 4l$ the pointed part $\mathcal{C}(sl(2), k)_{pt}$ is Tannakian. Let $A = [0] \oplus [k]$ be the regular etale algebra. It is known that the category $\mathcal{C}(sl(2), 4l)_A^0$ of dyslectic A -modules is simple. It is also known that this category is completely anisotropic if $l \neq 2, 7$. For $l = 2$ and $l = 7$ one has (see [DMNO])

$$(31) \quad [\mathcal{C}(sl(2), 8)] = [\mathcal{C}(sl(2), 3)_+]^{-2} = [\mathcal{C}(sl(2), 3)]^{-2} [\mathcal{C}(sl(2), 1)]^2,$$

$$(32) \quad [\mathcal{C}(sl(2), 28)] = [\mathcal{C}(sl(2), 3)_+] = [\mathcal{C}(sl(2), 3)][\mathcal{C}(sl(2), 1)]^{-1}.$$

Note also that $\mathcal{C}(sl(2), 4l)_A^0$ is pointed only for $l = 1$ and $\mathcal{C}(sl(2), 4)_A^0$ is of Witt order 4:

$$(33) \quad [\mathcal{C}(sl(2), 4)]^4 = 1.$$

For $k = 4l+2$ the pointed part $\mathcal{C}(sl(2), k)_{pt}$ is equivalent to $s\text{Vec}$. The centraliser $\mathcal{C}(sl(2), 4l+2)_+$ is a slightly degenerate category. By inspecting fusion rules it becomes clear that $\mathcal{C}(sl(2), 4l+2)_+$ is s -simple. It is non-trivial (i.e., does not coincide with $s\text{Vec}$) for $l > 0$. Triviality of $\mathcal{C}(sl(2), 2)_+$ corresponds to the fact that $\mathcal{C}(sl(2), 2)$ is an Ising category. In particular

$$(34) \quad [\mathcal{C}(sl(2), 2)]^{16} = 1.$$

It is also completely anisotropic for all l except $l = 2$ and it was shown in [DMNO] that

$$(35) \quad [\mathcal{C}(sl(2), 10)] = [\mathcal{C}(sl(2), 2)]^7.$$

For $l \neq 2$ we have two possibilities: either $\mathcal{C}(sl(2), 4l+2)_+$ is of infinite order in $s\mathcal{W}$ or $\mathcal{C}(sl(2), 4l+2)_+$ is braided equivalent to its reverse $\mathcal{C}(sl(2), k)_+^{rev}$. If $F : \mathcal{C}(sl(2), k)_+ \rightarrow \mathcal{C}(sl(2), k)_+^{rev}$ is a braided equivalence, then $F([j])$ is either $[j]$ or $[k-j]$ (this follows from the condition $\text{FPdim}(F([j])) = \text{FPdim}([j])$). Together

with the condition $\theta_{F([j])} = \theta_{[j]}^{-1}$ it gives $\theta_{[j]} = \pm \theta_{[j]}^{-1}$ for all $j = 2s$. This is equivalent to saying that $\frac{s(s+1)}{2l}$ is an integer for any $s = 0, \dots, 2l+1$. Clearly this can happen only for $l = 1$. Indeed, as was shown in [DMNO], one has the following relation between classes in \mathcal{W} :

$$(36) \quad [\mathcal{C}(sl(2), 6)]^2 = [\mathcal{C}(sl(2), 2)]^3,$$

which implies that the class of $\mathcal{C}(sl(2), 6)_+$ in $s\mathcal{W}$ has order 2. For all other values of l the class of the category $\mathcal{C}(sl(2), 4l+2)_+$ is of infinite order in $s\mathcal{W}$.

So far we have collected simple, completely anisotropic, non-degenerate categories

$$\begin{aligned} & \mathcal{C}(sl(2), 2l+1)_+, \quad l \geq 1, \\ & \mathcal{C}(sl(2), 4l)_A^0, \quad l \geq 3, \quad l \neq 7 \end{aligned}$$

and s-simple, completely anisotropic, slightly degenerate categories

$$\mathcal{C}(sl(2), 4l+2)_+, \quad l \geq 3.$$

By looking at the Frobenius-Perron dimensions we can see that all categories in the third series are different. It is easy to see by looking at the multiplicative central charges that the first two series have empty intersection and that all categories in the series are different. Moreover their corresponding slightly degenerate categories of the form $\mathcal{C} \boxtimes s\text{Vec}$ are s-simple, completely anisotropic and all different to each other. Finally none of them can appear in the third series (simply because in contrast to the categories in the third series their symmetric centers split out). Thus they are generators of the torsion free part of the subgroup of \mathcal{W} generated by classes $[\mathcal{C}(sl(2), k)]$.

Thus we have established the following.

Theorem 5.21. *All relations between the classes $[\mathcal{C}(sl(2), k)]$ in the Witt group \mathcal{W} follow from the relations (30 - 36).*

Proposition 5.22. *Let I be a finite subset of the set of odd positive integers. The category $\boxtimes_{k \in I} \mathcal{C}(sl(2), k)_+$ is completely anisotropic.*

Proof. It is known that categories $\boxtimes_{k \in I} \mathcal{C}(sl(2), k)_+$ are completely anisotropic [KiO]. By Corollary 5.10 it suffices to show that

$$\mathcal{C}(sl(2), k)_+ \not\cong (\mathcal{C}(sl(2), k')_+)^{\text{rev}}$$

for any odd $k \neq k'$. This is clear from looking at the Frobenius-Perron dimensions of categories $\mathcal{C}(sl(2), k)_+$. \square

REFERENCES

- [BaKi] B. Bakalov, A. Kirillov Jr., *Lectures on Tensor categories and modular functors*, AMS, (2001).
- [Be] R. Bezrukavnikov, *On tensor categories attached to cells in affine Weyl groups*, Representation theory of algebraic groups and quantum groups, 69–90, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.
- [BE] J. Bockenhauer, D. E. Evans, *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors*, Comm. Math. Phys. **213** (2000), no. 2, 267–289.
- [Da] A. Davydov, *Modular invariants for group-theoretical modular data*, J. Algebra, **323** (2010), 1321–1348.
- [DMNO] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, to appear in Journal für die reine und angewandte Mathematik (Crelle's Journal), eprint arXiv: 1009.2117 [math.QA] (2010).

- [De] P. Deligne, *Catégories tensorielles*, Moscow Math. Journal **2** (2002) no.2, 227 – 248.
- [DV] R. Dijkgraaf, E. Verlinde, *Modular invariance and the fusion algebras*, Nucl. Phys. (Proc. Suppl.) **5B** (1988), 87-97.
- [DGNO1] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik. *On braided fusion categories I*, Selecta Mathematica, **16** (2010), no. 1, 1 - 119.
- [DGNO2] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik. *On braided fusion categories II*, in preparation.
- [ENO1] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Annals of Mathematics **162** (2005), 581–642.
- [ENO2] P. Etingof, D. Nikshych, and V. Ostrik. *Weakly group-theoretical and solvable fusion categories*, Adv. Math., **226** (2011), 176–205.
- [ENO3] P. Etingof, D. Nikshych, and V. Ostrik. *Fusion categories and homotopy theory*, Quantum Topology, **1** (2010), no. 3, 209–273.
- [EO] P. Etingof, V. Ostrik, *Finite tensor categories*, Moscow Math. Journal **4** (2004), 627–654.
- [FK] I. Frenkel, V. Kac, *Basic representations of affine Lie algebras and dual resonansce models*, Invent. Math. **62** (1980), 23-66.
- [FFRS] J. Frohlich, J. Fuchs, I. Runkel, C. Schweigert, *Correspondences of ribbon categories*, Adv. Math. **199** (2006), no. 1, 192–329.
- [G] J. Greenough, Monoidal 2-structure of Bimodule Categories, J. Algebra **324**, no. 8, 1818–1859 (2010).
- [HuL] Y.-Z. Huang, J. Lepowski, *A theory of tensor products for module categories for a vertex operator algebra. I, II.*, Selecta Math. (N.S.) **1** (1995), no. 4, 699–756, 757–786.
- [JS] A. Joyal, R. Street, *Braided tensor categories*, Adv. Math., **102**, 20-78 (1993).
- [KiO] A. Kirillov Jr., V. Ostrik, *On q -analog of McKay correspondence and ADE classification of $\hat{\mathfrak{sl}}_2$ conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227.
- [KR] L. Kong, I. Runkel, *Morita classes of algebras in modular tensor categories*, Adv. Math. **219** (2008), no. 5, 1548–1576.
- [Ma] S. Majid, *Representations, duals and quantum doubles of monoidal categories*, Rend. Circ. Mat. Palermo (2) Suppl., **26** (1991), 197–206.
- [MS] G. Moore, N. Seiberg, *Naturality in conformal field theory*, Nucl. Phys. B **313** (1989), 16–40.
- [Mu1] M. Müger, *Galois theory for braided tensor categories and the modular closure*, Adv. Math. **150** (2000), no. 2, 151–201.
- [Mu2] M. Müger, *On the structure of modular categories*, Proc. Lond. Math. Soc., **87** (2003), 291–308.
- [Mu3] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Algebra **180** (2003), 81–157.
- [O1] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups, **8** (2003), 177–206.
- [Sch] P. Schauenburg, *The monoidal center construction and bimodules*, J. Pure Appl. Algebra **158** (2001), no. 2-3, 325–346.

A.D.: DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW HAMPSHIRE,
DURHAM, NH 03824, USA

E-mail address: alexei1davydov@gmail.com

D.N.: DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW HAMPSHIRE,
DURHAM, NH 03824, USA

E-mail address: nikshych@math.unh.edu

V.O.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403, USA
E-mail address: vostrik@math.uoregon.edu